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# THE MATHEMATICS STUDENT

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## FOURTEEN CALENDARS*


BY

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In these days of International Calendar Leagues bent upon uniformising the calendar, it may be heresy to dwell on the aesthetic character of the regularities underlying the apparent irregularities of the current calendar. The calendar now in use is based on the tropical year assumed for all practical purposes to consist of 365·25 days and four years are necessary to round off the fractional part. The rounding is done by the leap year which was made to come, by an ancient Imperial Ordinance, once in four years with an extra day. Since the Sun (or more precisely the Earth) cares very little for an Emperor, ancient or modern and has his (or her) own family troubles to mind, he (or she) chooses his (or her) own rate to travel, and we poor mortals have but to adjust ourselves to the celestial speed. When it was found that our rate was a little too much, it was proposed and adopted that we should drop three days in 400 years. This time the initiative came from a Pope. The century years came in handy for the lopping off, and those that are not divisible by 4 without the help of the last two ciphers were made to pay for their arithmetical privilege. Our prejudices regarding a year are thus based upon its relation to the number 4. Any number can behave with respect to 4 only in four different ways. It may leave a remainder 1, 2, or 3 when divided by 4, or be exactly divisible by it. The number 7 of our week-days is another arithmetical complicating factor. Still the irregularities are only finite in number, recurring generally in 28 years!

It is easily seen but not sufficiently well known that there are only *fourteen types* of calendars, *seven* ordinary and *seven* leap, which we may for convenience label on the analogy of vitamins:

(A, B, C, D, E, F, G) and (A′, B′, C′, D′, E′, F′, G′)

respectively, according as the first day of the year is a Sunday,

* The interested reader is advised to compare the rules given here with those outlined in the article 'The Elimination of Perpetual Calendars' by J. L. Roberts in *The Annals of Mathematical Statistics* pp. 44-46, Vol. VII 1936.

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Monday, Tuesday, Wednesday, Thursday, Friday, or a Saturday. We shall, now, distinguish three *species* of ordinary years as $O_1$, $O_2$, $O_3$ according as they leave remainder 1, 2, or 3 when divided by 4 and give the following rule for the recurrence of the calendars *in the same century*.

*Write the numbers 6, 11, 11 round a circle and mark against them $O_1$, $O_2$, $O_3$ respectively in the anti-clockwise direction. Going forward in time, you will note that the first period of recurrence is the number marked against it and the other periods follow in the clockwise cyclic order. While reckoning backwards in time, the principle is that the $O_2$ species recurs backwards and forwards in the same manner, while the forward recurrence-periods for $O_1$ are the same as the backward recurrence-periods for $O_3$ and vice-versa. The leap-year recurs only in 28 year periods, backwards or forwards.*

Applying the above rules, we easily verify that the current calendar for 1937 is identical with those of

1909, 1915, 1926, 1943, 1954, 1965, 1971, 1982, 1993, 1999

in the present century.

A note of caution! Since a century year is not considered ordinary or leap in the same way as the other years, the aforesaid principles will not work if a century year intervenes, and they become applicable for any century provided we know the type and species of the first year of the century. The law of succession of the types in any century is given by the following cycle:

ABCD′ | FGAB′ | DEFG′ | BCDE′ | GABC′ | EFGA′ | CDEF′ |
which contains seven groups of 4 letters.

The letters of a group are associated, in order, with the three ordinary years $O_1$, $O_2$, $O_3$ and a leap year. The letters (ABCDEFG) occur cyclically except after a leap year indicated by a dash, the leap year letter being followed by the alternate letter in the cycle (for example D′ is not followed by E but by F). Further in the entire cycle, a letter without a dash occurs once and once only as the first, second, or third letter in the four-letter group.

It is useful to note that all centuries begin as an $O_1$—year and the type of the first year of any century can be determined by noting the type and character (ordinary or leap) of the previous century year. For example, the next century year 2000 being leap, our arithmetic and alphabetic cycles will work right up to 2100 A.D. Since

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2100 is an ordinary year, its type determined by our rules must drop the dash. Now 1937 is of the F-type in the $O_1$ – species and the alphabetic cycle starting with FGAB′ passes in its sixth course through 2100, and the letter corresponding to it is the 24th from F, viz. F′ which must now drop the dash. Thus 2100 repeats the calendar of the current year 1937! The twenty-second century therefore begins with the G-type in the $O_1$ – species and the alphabetic cycle for the century runs

GABC′ | EFGA′ | · · · .

In the present depression it may not be wise to waste enormous time and money in printing calendars from year to year, when what we require is after all *Fourteen Standard Calendars*, for instance, those of the years.

| | | | |
|---|---|---|---|
| 1901 (C), | 1902 (D), | 1903 (E), | 1904 (F′) |
| 1905 (A), | 1906 (B), | 1908 (D′) | 1909 (F), |
| 1910 (G), | 1912 (B′), | 1916 (G′) | 1920 (E′), |
| 1924 (C′), | 1928 (A′) | | |

which will repeat now and again according to the laws indicated above, at least for a millennium, provided no reforms of the calendars come into general use in this period.

## Addendum

If we define the type of a century as the type of the first year of the century, we have according to the original calendar system known as the Old Style, only 7 types A—G and they occur cyclically in the alphabetic or the reverse order according as we proceed backwards or forwards in time. The transition century in which the change from the Old to the New Style took place, has to be assigned two types, one corresponding to each Style, as per rule explained below by an example. The year 1752 when the New Style began to be adopted in the British Dominions by giving the date September 14 to the day after September 2, belongs to a century of the G-type (N.S.) and the type (O.S.) of the same century is read as D from the cycle A—G, by counting 12 places (the difference between the two dates) from G, (which is the same thing as counting four places from A). We are now in a position to read off from the cycle D E F G A B C, the types of all the centuries prior to the 18th until the century when the Old Style came into vogue.

In the New Style, however, there are only 4 types of centuries and they follow the cyclic order of succession C B G E, the type of the twentieth century being C.

# NOTE ON SCHERING'S THEOREM ON THE INTEGRALS OF A DYNAMICAL SYSTEM

BY

H. S. SUBBA RAO, *Research Scholar*, *University of Mysore.*

**1.** *Introduction.*

If the Hamiltonian H of a dynamical system be explicitly independent of the time $t$, and if

$$\phi(p_1, \ldots p_n, q_1, \ldots q_n, t) = \text{constant}$$

be a time-dependent integral, it is well known* that

$$\frac{\partial \phi}{\partial t} = \text{const.}, \; \frac{\partial^2 \phi}{\partial t^2} = \text{const., etc.,}$$

are also integrals. The question naturally arises whether the integrals thus obtained by successive partial differentiations are all independent and if it be possible to obtain $2n$ independent integrals of the system in this manner. Since it is known† that $2n$ independent integrals form a function group, it is at once obvious that this process of differentiation if continued indefinitely cannot yield all independent integrals. It is also possible that the differentiation may stop at a certain stage yielding less than $2n$ independent integrals, or more, but not all independent. The several possibilities envisaged depend on the form of the function $\phi$, and it would be impossible to derive definite conclusions without a knowledge of this form. In this paper I have considered polynomial integrals and proved the theorem that, if $\phi = \text{const.}$ be considered an independent integral, it cannot be a polynomial in $t$ unless it be a linear function. Thus the case considered is one in which the differentiation stops after the first stage.

**2.** *Complete set of integrals related to $H = h$.*

Since the Hamiltonian does not contain $t$ explicitly, the integral

$$\text{H} = h \qquad \ldots \quad (1)$$

where $h$ is a constant gives the integral of energy. By the use of this integral we can effect a reduction of the order‡ of the system.

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* E. Schering. *Gött Abh.* 19 (1874), p. 3.
Whittaker, *Analytical Dynamics* (3rd Edition) p. 336, Ex. 2.

† See Prange: *Ency. Math. Wiss.* Bd. 4, Heft. 4. § 26, p. 704.

‡ See Whittaker *ibid*, § 141, p. 313.

The original canonical equations are replaced by

$$\left.\begin{aligned} &\frac{dq_r}{dt}=\frac{\partial K}{\partial p_r},\ \frac{dp_r}{dt}=-\frac{\partial K}{\partial q_r} \qquad (r=2, 3, \dots n) \\ &\frac{dt}{dq_1}=\frac{\partial K}{\partial h},\ \frac{dh}{dq_1}=0 \end{aligned}\right\} \quad \dots \quad (2)$$

where the equation $H=h$ solved for $p_1$ is written

$$p_1=K\,(p_2, p_3, \dots, p_n, q_1 \dots q_n, h).$$

The first $(2n-2)$ equations of (2) are the canonical equations of a system of order $2n-2$, the last equation expresses the fact that $h$ is a constant and the last but one gives the integral

$$\int \frac{\partial K}{\partial h}\, dq_1 = \int dt + \text{const.}$$

*i.e.* an integral of the form

$$\theta = \theta_0 + t = \text{const.} \qquad .. \quad (3)$$

($\theta_0$ being independent of $t$).

(3) is obviously conjugate to (1) since it is easy to verify that

$$(H, \theta) = 1.$$

If $r_1, r_2, \dots r_{2n-2}$ be independent integrals of the reduced system we can also prove that

$$(H, r_s)=0, \qquad (\theta, r_s)=0, \qquad (s=1, 2, \dots 2n-2)$$

The integrals $\{H, \theta\}$ and $\{r_1, \dots r_{2n-2}\}$ form two sub-groups of the total function group of order $2n$, and are moreover reciprocal to each other** and constitute a complete system of integrals. Also $r_s$ is independent of $t$.

If $\phi=c$ be a time-dependent integral, we should have

$$\phi = \phi\,(H, \theta, r_1 \dots r_{2n-2})$$

so that

$$\frac{\partial\phi}{\partial t}=\frac{\partial\phi}{\partial\theta}\cdot\frac{\partial\theta}{\partial t}=\frac{\partial\phi}{\partial\theta}$$

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\partial^2\phi}{\partial\theta^2}, \text{ etc.}$$

This shows that $\phi, \frac{\partial\phi}{\partial t}, \frac{\partial^2\phi}{\partial t^2}$, etc. all belong to the function group set up above and cannot all be independent even if $\phi$ be infinitely differentiable.

** See Prange *ibid*, p. 720, foot-note (351).

**3.** *Complete set of integrals related to $\phi = c$.*

In the above article we have not treated $\phi = c$ as an independent integral of a complete system to which it belongs, but as belonging to another basis of the function group. In an entirely analogous manner we can set up a complete system of integrals given by the two sub-groups $(\phi, \chi)$ and $(\alpha_1, \ldots \alpha_{2n-2})$ reciprocal to each other such that $\chi$ is conjugate to $\phi$. We therefore have

$$(\phi, \chi) = 1 \; ; \; (\phi, \alpha_r) = 0, \; (\chi, \alpha_r) = 0 \quad (r = 1, 2 \ldots 2n - 2) \qquad \ldots \quad (4)$$

This complete system in which $\phi$ is treated as an independent integral enables us to derive a special theorem.

**4.** *Polynomial integral in t.*

We shall now prove the theorem that $\phi$, if treated as an independent integral of a complete system, cannot be a polynomial in $t$ of degree higher than one.

Let
$$\phi = a_0 + a_1 t + \cdots + a_h t^h, \qquad \ldots \quad (5)$$

where $a_0, a_1, \ldots a_k$ are all functions of the co-ordinates and momenta. Denoting $\frac{\partial \phi}{\partial t}, \frac{\partial^2 \phi}{\partial t^2}$, etc. by $\phi_1$, $\phi_2$, etc. we have, since $\phi_1$, $\phi_2$, etc. are all integrals,

$$(\phi, \mathrm{H}) = \phi_1, \quad (\phi_1, \mathrm{H}) = \phi_2, \; \ldots \; (\phi_{h-1}, \mathrm{H}) = \phi_h. \qquad \ldots \quad (6)$$

Equating like powers of $t$ on the two sides of each of these equations we have the relations,

$$(\mathrm{H}, a_0) = a_1, \; (\mathrm{H}, a_1) = 2a_2, \ldots (\mathrm{H}, a_{h-1}) = k a_h, \; (\mathrm{H}, a_h) = 0 \qquad \ldots \quad (7)$$

The last equation of (7) shows that $a_k$ = constant is itself an integral. Substituting (5) in (4) and equating like powers of $t$, we have

$$\left. \begin{array}{l} (a_0, \chi) = 1, \; (a_1, \chi) = (a_2, \chi) = \cdots\cdots = (a_h, \chi) = 0 \\ (a_0, \alpha_r) = (a_1, \alpha_r) = \cdots\cdots\cdots\cdots = (a_k, \alpha_r) = 0 \end{array} \right\} \qquad \ldots \quad (8)$$

Since $a_k$ = constant is an integral and $(a_h, \alpha_r) = 0$ it follows that it belongs to the first sub-group of order 2; but from $(a_k, \chi) = 0$ we deduce that either $a_h$ is identically a constant or a function of $\chi$ only. The latter case is ruled out since $\phi$ and $\chi$ are independent integrals. Therefore $a_h$ is identically a constant. We can further prove that this constant is zero, as follows:

Since $\phi_{h-1}$ = constant is also an integral we have $a_{h-1} + k a_h t$ = constant, an integral.

In virtue of relations (8) we can deduce

$$(\phi_{k-1}, \alpha_r)=0, \text{ and } (\phi_{k-1}, X)=0$$

and reasoning just as above we can show that $\phi_{k-1}$ is identically equal to a constant; but this is true only if $a_k=0$ and $a_{k-1}$ identically equal to a constant.

Proceeding now with $\phi_{k-2}$ we can prove that $a_{k-1}=0$ and $a_{k-2}\equiv$constant.

In a finite number of steps we deduce that $\phi$ has the form

$$\phi = a_0 + a_1 t \qquad \dots \quad (9)$$

wherein we can show, as in (7) that $a_1=$constant is an integral, and further that $a_1$ is identically a constant. In virtue, however, of the first equation of (8) viz. $(a_0, X)=1\neq 0$, we cannot proceed further and deduce $a_1=0$. (9) is therefore the ultimate possible form of $\phi$ and this proves the theorem. This excludes the possibility of the differentiations stopping after a finite number of steps.

**5.** *Conclusion.* I hope to return in a further paper to other special cases specially the case of integrals periodic in $t$ in which the function repeats itself after two partial differentiations.

I wish to thank Prof. B. S. Madhava Rao for the suggestion of the problem and kind guidance.

# NUMERICAL IDENTITIES *

BY

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I. (α) By means of the formula

$$2(x^2+y^2)^2=(x^2-2xy-y^2)^2+(x^2+2xy-y^2)^2$$

we obtain identities of the form $a^2+a^2=b^2+c^2$.

(β) If we apply to these identities the theorem which I have given in *The Mathematics Student*, Vol. II, page 102 (II), we obtain the relation †

$$(t-a,\ t-a,\ t+a,\ t+a)^n=(t-c,\ t-b,\ t+b,\ t+c)^n \qquad n=1,2,3$$

where $t$ is any integer.

Example: Take $x=3$, $y=1$ in (α) so that $5^2+5^2=1^2+7^2$. If $t=9$ we get

$$(4,\ 4,\ 14,\ 14)^n=(2,\ 8,\ 10,\ 16)^n \qquad n=1,2,3.$$

II. From the relation

$$(\mathrm{A},\ \mathrm{A},\ \mathrm{B},\ \mathrm{B})^n=(\mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F})^n \qquad n=1,2,3$$

we obtain a similar relation in which each of the numbers A, B, C, D, E, F is replaced by $q\mathrm{A}\pm r$, $q\mathrm{B}\pm r$, $q\mathrm{C}\pm r$, $q\mathrm{D}\pm r$, $q\mathrm{E}\pm r$, $q\mathrm{F}\pm r$ where $q$ and $r$ are any integers. If we are able to express $q\mathrm{B}-q\mathrm{A}$ as the product of two factors $m$ and $p$ which are both odd or both even, we have then

$$q\mathrm{A}+r=\mathrm{X}^2; \qquad q\mathrm{B}+r=\mathrm{Y}^2$$

where

$$\mathrm{X}=\tfrac{1}{2}(m-p); \quad \mathrm{Y}=\tfrac{1}{2}(m+p); \quad r=\mathrm{X}^2-q\mathrm{A}=\mathrm{Y}^2-q\mathrm{B}$$

and thus we get solutions in integers of the equations

$$(\mathrm{X},\ \mathrm{X},\ \mathrm{Y},\ \mathrm{Y})^{2n}=(\mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F})^n \qquad n=1,2,3.$$

Example:

$$(9,\ 9,\ 11,\ 11)^{2n}=(73,\ 97,\ 105,\ 129)^n \qquad n=1,2,3$$
$$(3,\ 3,\ 7,\ 7)^{2n}=(1,\ 25,\ 33,\ 57)^n \qquad n=1,2,3.$$

III. Identities of the form

$$2(\mathrm{P}_1,\ \mathrm{P}_2,\ \mathrm{P}_3)^{2n}=(\mathrm{R}_1,\ \mathrm{R}_2,\ \mathrm{R}_3 \ldots \mathrm{R}_6)^n \qquad n=1,2,3,4,5$$

may be obtained by the following method.

---

* (Translated from the original in German. ED.)

† $(a,\ b,\ c\ \ldots)^n$ means $a^n+b^n+c^n+\cdots$

($\alpha$) If it is given that $a^2+b^2=c^2$, then $(a+b)^2-c^2=c^2-(b-a)^2$ $=s$ (say). Now if we have $(a, a)^n = (\mathrm{M, N, Q})^n \quad n=2, 4$ and if $s$ is divisible by $a$, so that $s=at$ we may multiply this equation by $t^n$ and use the result II on page 102 of my paper cited earlier with $\mathrm{T}=c^2$.

We thus obtain

$$2(c^2-s,\ c^2,\ c^2+s)^n=(c^2-\mathrm{Q}t,\ c^2-\mathrm{N}t,\ c^2-\mathrm{M}t,\ c^2+\mathrm{M}t,\ c^2+\mathrm{N}t,\ c^2+\mathrm{Q}t)^n$$
$$n=1, 2, 3, 4, 5.$$

Now $c^2-s=(b-a)^2$, and $c^2+s=(b+a)^2$ so that the left side is

$$2(b-a,\ c,\ b+a)^{2n}$$

and so we have an identity of the desired form.

Example: $7^2+24^2=25^2$ and accordingly $(24-7)^2, 25^2, (24+7)^2$ are in A. P. with the common difference $s=336$. Also $(7, 7)^n=(3, 5, 8)^n$ for $n=2, 4$. Here $s/a=t=48$, $c=25$ and we have

$$2(17, 25, 31)^{2n}=(241, 385, 481, 769, 865, 1009)^n$$
$$n=1, 2, 3, 4, 5.$$

Similarly from $13^2+84^2=85^2$ and $13^n+13^n=7^n+8^n+15^n$ for $n=2, 4$ we get

$$2(71, 85, 97)^{2n}=(4705, 5881, 6049, 8401, 8569, 9745)^n$$
$$n=1, 2, 3, 4, 5.$$

IV. If we are given that

$$(\mathrm{A}_1, \ldots \mathrm{A}_8)^n=(\mathrm{B}_1, \ldots \mathrm{B}_8)^n \qquad n=1, 2, 3, 4, 5, 6$$

(vide *The Mathematics Student*, Vol. II, p. 103) and if we have

$$\mathrm{A}_1+\mathrm{B}_8=\mathrm{A}_2+\mathrm{B}_7=\cdots\ \mathrm{A}_8+\mathrm{B}_1=\mathrm{K}\ \text{(say)}$$

and if $s=\frac{1}{2}\mathrm{K}$, then

$$(\mathrm{A}_1-s, \ldots\ \mathrm{A}_8-s)^n=(\mathrm{B}_1-s, \ldots\ \mathrm{B}_8-s)^n \qquad n=1, 2 \ldots 6$$

whence we get identities of the form

$$(-\mathrm{C}_1,\ -\mathrm{C}_2,\ -\mathrm{C}_3,\ -\mathrm{C}_4,\ \mathrm{D}_1,\ \mathrm{D}_2,\ \mathrm{D}_3,\ \mathrm{D}_4)^n=$$
$$(-\mathrm{D}_4,\ -\mathrm{D}_3,\ -\mathrm{D}_2,\ -\mathrm{D}_1,\ \mathrm{C}_4,\ \mathrm{C}_3,\ \mathrm{C}_2,\ \mathrm{C}_1)^n$$
$$n=1, 2 \ldots 6$$

whence $\qquad (\mathrm{C}_1\ \mathrm{C}_2\ \mathrm{C}_3\ \mathrm{C}_4)^n=(\mathrm{D}_1\ \mathrm{D}_2\ \mathrm{D}_3\ \mathrm{D}_4)^n \qquad n=1, 3, 5.$

Thus from the last example on page 103 (*The Mathematics Student*, Vol. II) we have on multiplication by $2^n$

$$(4, 4, 16, 22, 26, 38, 42, 48)^n=(2, 8, 12, 24, 28, 34, 46, 46)^n$$
$$n=1, 2, \ldots 6.$$

and we have $\qquad 4+46=16+34=\cdots\ 48+2=\mathrm{K}.$

We thus obtain

$$(3,\ 9,\ 21,\ 21)^n=(1,\ 13,\ 17,\ 23)^n \qquad n=1,\ 3,\ 5.$$

In the same way we may obtain identities involving a larger number of terms

$$(23,\ 33,\ 55,\ 57)^n = (1,\ 5,\ 13,\ 39,\ 51,\ 59)^n \qquad n=1,\ 3,\ 5,\ 7$$

$$(1,\ 13,\ 25,\ 55,\ 75,\ 87,\ 95)^n=(7,\ 11,\ 19,\ 61,\ 69,\ 91,\ 93)^n \quad n=1,\ 3,\ 5,\ 7,\ 9$$

$$(1,\ 5,\ 7,\ 17,\ 26,\ 27,\ 30,\ 37,\ 38,\ 50,\ 51,\ 54,\ 58)^n=$$
$$(16,\ 20,\ 20,\ 24,\ 32,\ 36,\ 43,\ 45,\ 52,\ 56,\ 57)^n$$
$$n=1,\ 3,\ 5,\ 7,\ 9,\ 11.$$

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# TWO CURVES ASSOCIATED WITH A TETRAHEDRON

BY

N. DURAIRAJAN

1. $A_1\,A_2\,A_3\,A_4$ is the fundamental Tetrahedron. The locus of a point P such that three pairs of planes $PA_1\,A_2$, $PA_3\,A_4$ etc. are perpendicular is a quartic curve in space.

Let the rectangular Cartesian equations of the four faces be

$$F_r=l_r\ x+m_r\ y+n_r\ z+p_r=0 \qquad (r=1,\ 2,\ 3,\ 4).$$

The condition that the planes $F_1$ and $F_2$ are perpendicular is

$$l_1\,l_2+m_1\,m_2+n_1\,n_2=0$$

and this may be written in the form

$$0=\psi\,(F_1\,F_2)=l_1\,l_2\ +\ m_1\,m_2\ +n_1\,n_2=\frac{\partial F_1}{\partial x}\cdot\frac{\partial F_2}{\partial x}+\frac{\partial F_1}{\partial y}\,\frac{\partial F_2}{\partial y}+\frac{\partial F_1}{\partial z}\,\frac{\partial F_2}{\partial z}$$

If the co-ordinate of P be $(x'y'z')$, the equation of the planes $PA_1A_2$, $PA_3A_4$, are $F_3\,F_4'-F_4\,F_3'=0$ and $F_1\,F_2'-F_2\,F_1'=0$ where $F'_r=F_r\,(x'y'z')$. These are perpendiculars if

$$\psi\,(F_1\ F_3)\,F_2'\ F_4'+\psi\,(F_2\ F_4)\,F_1'\ F_3'=\psi\,(F_1\ F_4)\,F_2'\ F_3'+\psi\,(F_2\ F_3)\,F_1'\ F_4'.$$

Hence the locus we are seeking is given by

$$\begin{aligned}\psi\,(F_1\,F_3)\,F_2\,F_4+\psi\,(F_2\,F_4)\,F_1\,F_3\\ =\psi\,(F_1\,F_4)\,F_2\,F_3+\psi\,(F_2\,F_3)\,F_1\,F_4\\ =\psi\,(F_1\,F_2)\,F_3\,F_4+\psi\,(F_3\,F_4)\,F_1\,F_2\end{aligned} \qquad \dots \quad (1)$$

The curve is the common intersection of two quadrics.

**2.** The curve passes through the four vertices and the four feet of the altitudes of the Tetrahedron.

It is obvious that the vertices are points on the curve.

The altitude from $A_1$ is given by

$$\left\| \begin{matrix} F_2 & F_3 & F_4 \\ \psi(F_1\, F_2) & \psi(F_1\, F_3) & \psi(F_1\, F_4) \end{matrix} \right\| = 0 \qquad \ldots \quad (2)$$

Hence the points given by

$$F_1 = 0 \quad \text{and} \quad \frac{F_2}{\psi(F_1\, F_2)} = \frac{F_3}{\psi(F_1\, F_3)} = \frac{F_4}{\psi(F_1\, F_4)}$$

lies on the curve, and this is the foot of one of the altitudes.

**3.** The cosine of the interior angles between the faces of the tetrahedron may be denoted by $c_{12}\, c_{13}$ etc. corresponding to the pairs of faces 1, 2 ; 1, 3 ; etc., so that

$$c_{12} = (l_1\, l_2 + m_1\, m_2 + n_1\, n_2) / \sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}.$$

The locus (1) may therefore be written in the form

$$\begin{aligned} x_4\, x_1 \cdot c_{23} + x_2 x_3 \cdot c_{14} \\ &= x_4 x_2 \cdot c_{31} + x_3 x_1 \cdot c_{24} \\ &= x_4 x_3 \cdot c_{12} + x_1 x_2 \cdot c_{34} \qquad \ldots \quad (3) \end{aligned}$$

where $x_1\, x_2\, x_3\, x_4$ are the normal co-ordinates of P with respect to the tetrahedron.

**4.** The four altitudes lie on the " Associated Hyperboloid ", whose equation* is

$$\begin{aligned} &(c_{41} \cdot c_{23} - c_{42} \cdot c_{31})\,(x_4 x_2 \cdot c_{13} + x_3 x_1 \cdot c_{42}) \\ &+ (c_{42} \cdot c_{31} - c_{43} \cdot c_{12})\,(x_4 x_1 \cdot c_{23} + x_2 x_3 \cdot c_{41}) \\ &+ (c_{43} \cdot c_{12} - c_{41} \cdot c_{23})\,(x_4 x_3 \cdot c_{12} + x_1 x_2 \cdot c_{43}) = 0 \qquad \ldots \quad (4) \end{aligned}$$

Hence the curve lies on the Associated Hyperboloid.

**5.** The curve degenerates when one or more pairs of opposite edges are perpendicular. When the tetrahedron is orthocentric, the curve consists of the four lines joining the vertices to the orthocentre. Other cases of degeneration may be worked out; for example when $c_{41} = 0$ or when $c_{41} = c_{23} = 0$.

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* Please see J. I. M. S. Vol. 5, p. 54, where the equation in tetrahedral co-ordinates is obtained by A. C. L. Wilkinson.

**6.** Consider the locus of a point Q which is such that its pedal tetrahedron $B_1 B_2 B_3 B_4$ is orthocentric. We have then

$$B_1B_2^2 + B_3B_4^2 = B_1B_3^2 + B_2B_4^2 = B_1B_4^2 + B_2B_3^2$$

that is

$$x_1^2 + x_2^2 + 2x_1x_2c_{12} + x_3^2 + x_4 + 2x_3x_4c_{34} = 2 \text{ similar expressions.}$$

$$\therefore\ x_4x_1 \cdot c_{41} + x_2x_3 \cdot c_{23} = x_4x_3 \cdot c_{43} + x_1x_2 \cdot c_{12} = x_4x_2 \cdot c_{42} + x_3x_1c_{13} \quad \ldots (5)$$

**7.** Cases of degeneracy of the locus (5) may also be worked out. If the given tetrahedron is orthocentric, the curve splits up into four lines joining the circumcentre to the four vertices.

The curve (5) is thus the isogonal transform of the locus (3) in the isogonal transformation

$$X_1 : X_2 : X_3 : X_4 = x_2\, x_3\, x_4 : x_1\, x_3\, x_4 : x_1\, x_2\, x_4 : x_1\, x_2\, x_3$$

with respect to the tetrahedron ABCD.

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# EFFECT OF DERANGEMENT OF TERMS ON SOME SEMI-CONVERGENT SERIES

BY

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**1.** The theory of derangement of terms of semi-convergent series has been studied by Riemann and Pringsheim.* In this paper an attempt is made to study the problem independently when we have $nf(n) \to 0$ where $n \to \infty$.

**2.** Notation :—Let

$\sum_n$ = sum of the first $n$ terms of the original series,

$\sum'_n$ = ,, ,, ,, deranged series,

and $T_{pq} = f(2q+1) + f(2q+3) + \ldots\ldots\ldots + f(2p-1)$

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* Cf. Bromwich, *Infinite Series*.

If $f(x)$ is a steadily decreasing positive function tending to $o$, the series $\Sigma(-1)^{n-1} f(n)$, where $n$ is a positive integer, converges ; and that

(i) if $nf(n)$ tends steadily to $\infty$ with $n$, the values of $p$ requisite for an alteration $l$ in the sum of the series is subject to the condition $\text{Lim}\,(p-q)\,f(2q) = l$, when $p, q \longrightarrow \infty$, $p$ and $q$ being the number of positive and negative terms in the first $p+q$ terms of the deranged series;

(ii) if $\lim nf(n) = g$, and if $k$ is the limit of the ratio of the number of positive terms to the number of negative terms, the alteration $l$ is given by $2l = g \log k$

Pringsheim had also studied the case when $\text{Lim}\; nf(n) = o$.

where $p$ is the number of positive terms and $q$ the number of negative terms in the first $p+q$ terms of the deranged series.

$p-q$ is considered positive unless otherwise mentioned.

**3.** THEOREM I. *For any finite alteration in the sum of a semi-convergent series of the form $\sum(-1)^{n-1} f(n)$ where $nf(n)\to 0$ as $n\to\infty$ by derangement of its terms,*

(i) $\overline{\operatorname{Lim}}_{p,\,q\to\infty} p/q$ must be $\infty$, *if* $p>q$ and

(ii) $\overline{\operatorname{Lim}}_{p,\,q\to\infty} q/p$ must be $\infty$, *if* $q>p$ †

$\sum'_{p+q} = \sum_{2q} + T_{pq}$.

Now, $nf(n)\to 0$ as $n\to\infty$, so that $0< xf(x) <\varepsilon$ if $x>n_0(\varepsilon)$.

If both $p$ and $q$ be chosen greater than $n_0(\varepsilon)$, we have when $p\to\infty$, $q\to\infty$

$$\overline{\lim}\ T_{pq} < \overline{\lim}\left[\ \varepsilon\ \left\{\frac{1}{2q+1}+\frac{1}{2q+3}+\ldots+\frac{1}{2p-1}\right\}\right]$$

$$=\overline{\lim}\left[\frac{\varepsilon}{2}\left(\frac{2}{2q+1}+\frac{2}{2q+3}+\ldots.+\frac{2}{2p-1}\right)\right]$$

But $\dfrac{2}{2q+1}\cdots\dfrac{2}{2p-1}<\dfrac{1}{q}+\dfrac{1}{q+1}+\cdots+\dfrac{1}{p-1}=\log\dfrac{p-1}{q}+\eta,$

where $\eta\to 0$ as $p,\to\infty$.

$$\therefore\quad \overline{\lim}\ T_{pq}\leqslant\overline{\operatorname{Lim}}\frac{\varepsilon}{2}\left[\log\frac{p-1}{q}+\eta\right]$$

$$\therefore\quad T=\overline{\lim}\ T_{pq}=o,\ \text{if } p/q \text{ is bounded as } p,\ q\to\infty.$$

In this case $\overline{\lim}\sum'_{p+q}=\overline{\lim}\sum_{2q}=S$, the sum of the original series.

Hence, for any finite alteration in the sum of the deranged series $\overline{\operatorname{Lim}}_{p,\,q\to\infty} p/q=\infty$. Similarly the second part can be proved.

**4.** THEOREM II. *If $n\log n\, f(n)\to g$ as $n\to\infty$, then, for an alteration not exceeding $T$ in the sum of the deranged series, $2p=O\,[(2q)^{t+\varepsilon}]$ where $t=e^{T/g}$. where $\varepsilon>0$ and may be as small as we please.*

We have here $g-\varepsilon< x\log x\, f(x)< g+\varepsilon$ *if* $x>n_1(\varepsilon)$.

† Results similar to case (ii) for all other theorems.

Now if $p$ and $q$ be chosen greater than $n_1$, $T_{pq}$ lies between $(g-\varepsilon)$ $R_{p,q}$ and $(g+\varepsilon)$ $R_{pq}$ where

$$R_{pq}=\left[\frac{1}{(2q+1)\log(2q+1)}+\cdots\cdots\cdots+\frac{1}{(2p-1)\log(2p-1)}\right]$$

But $R_{p,q}$ lies between

$$\int_{2q-1}^{2p-1}\frac{dx}{x\log x}=\log\frac{\log(2p-1)}{\log(2q-1)}\text{ and }\int_{2q+1}^{2p+1}\frac{dx}{x\log x}=\frac{\log(2p+1)}{\log(2q+1)}$$

$\therefore$ If $\overline{\lim\limits_{p,q\to\infty}}\dfrac{\log 2p}{\log 2q}=t$ (a finite number), both the integrals tend to the limit $\log t$ as $p$ and $q\to\infty$. Hence, it follows that

$$T=\overline{\lim_{p,q\to\infty}}\;T_{p,q,}=g\log t$$

and $\dfrac{\log 2p}{\log 2q}>t+\varepsilon$ where $\varepsilon\to o$ as $p$ and $q\to\infty$ and is positive that is

$$2p=O\left[(2q)^{t+\varepsilon}\right]\text{ where }t=e^{T/g}.$$

5. If $n\log n\,f(n)$ tends steadily to $\infty$, it is not possible to define a regular relationship between $p$ and $q$ such that the alteration in the sum of the deranged series is a finite number, unless more definite information about $n\log n\,f(n)$ is known.

Since $p/q\to\infty$ with $p$ and $q$, $\text{Lim}\,(2q\log 2q)/2p\log(2p)=0$; again since $nf(n)$ decreases steadily to 0, $2p\,f(2p)<2q\,f(2q)$ and since $n\log n\,f(n)$ increases steadily to $\infty$,

$$2p\log(2p)\;f(2p)>2q\log(2q)\;f(2q)$$

Thus $$\frac{q\log(2q)}{p\log(2p)}<\frac{f(2p)}{f(2q)}<\frac{q}{p}$$

But both $\dfrac{q}{p}$ and $\dfrac{q}{p}\dfrac{\log(2q)}{\log(2p)}\to 0$ as $p$ and $q\to\infty$ so that

$$f(2p)/f(2q)\to 0\text{ as }p,\;q\to\infty.$$

Now

$$(p-q)\,f(2p)<T_{pq}<(p-q)\,f(2q)$$

$T_{pq}$ may oscillate infinitely when $p,q\to\infty$.

But the following is an immediate consequence of the above.

THEOREM III. *If $p-q$ is taken to be the integer nearest to $l/f(2q)$, where $l$ is a preassigned finite number, the deranged series has either a finite sum, or oscillates finitely.*

Here $(p-q)\, f(2q) = l + \text{F}.\, f(2q)$ when F is a proper fraction and $(p-q)\, f(2q) \to l$ when $p, q \to \infty$.

Hence $\lim\, (p-q)\, f(2p) = \text{Lim}\, (p-q)\, f(2q). \dfrac{f(2p)}{f(2q)} = l \times 0 = 0$

We may observe that if $p-q =$ the integer nearest to $l/f(2p)$, where $l$ is a finite number, the deranged series may oscillate infinitely.

6. THEOREM IV. *If $n \log (\log n)\, f(n) \to g$, where $g$ is finite, the deranged series becomes divergent, provided $p > q$.*

If $p$ and $q$ are chosen sufficiently large, $\text{T}_{pq}$ lies between $(g-\varepsilon)\, \text{R}_{pq}$ and $(g+\varepsilon)\, \text{R}_{pq}$ where

$$\text{R}_{qq} = \frac{1}{(2q+1) \log \{\log (2q+1)\}} + \frac{1}{(2q+3) \log \{\log (2q+3)\}} + \dots\dots\dots + \frac{1}{(2p-1) \log \{\log (2p-1)\}}\,.$$

But $\text{R}_{pq}$ lies between $\text{I}_1 = \displaystyle\int_{2q-1}^{2p-1} \frac{dx}{x \log (\log x)}$ and $\text{I}_2 = \displaystyle\int_{2q+1}^{2p+1} \frac{dx}{x \log (\log x)}$

Hence $(g-\varepsilon)\, \text{I}_2 < \text{T}_{pq} < (g+\varepsilon)\, \text{I}_1$

Again, if $\text{I} = \displaystyle\int_{2q}^{2p} \frac{dx}{x \log (\log x)} = \int_{\log (\log 2q)}^{\log (\log 2p)} \frac{e^x\, dx}{x}$, then since

Since $e^s \displaystyle\int_s^l \frac{dx}{x} < \int_s^l \frac{e^x}{x}\, dx < e^l \int_s^l \frac{dx}{x}$,

I lies between $\log (2q). \log \left\{ \dfrac{\log (\log 2p)}{\log (\log 2q)} \right\}$ and $\log 2p. \log \left\{ \dfrac{\log (\log 2p)}{\log (\log 2q)} \right\}$

But $\log \dfrac{\log (\log 2p)}{\log (\log 2q)}$ is not less than 1, since $p<q$ and as $p$ and $q$ tend to infinity, both $\log p$ and $\log q$ tend to infinity.

Hence $\text{I} \to \infty$ with $p$ and $q$, $\therefore$ $\text{T}_{pq} \to \infty$ and the deranged series is divergent.

**7.** THEOREM V. *If $n \log n\, f(n) \to 0$ and if $p = O(q^t)$ when $t$ is a finite number, the sum of the series will be unaltered by derangement.*

If $p$ and $q$ are chosen sufficiently large,

$$T_{pq} < \varepsilon \left[ \frac{1}{2q \log (2q)} + \frac{1}{(2q+2)\log(2q+2)} + \cdots + \frac{1}{2p \log (2p)} \right]$$

$$< \frac{\varepsilon}{2} \left[ \frac{2}{2q \log (2q)} + \int_{2q}^{2p} \frac{dx}{x \log x} \right]$$

$$= \frac{\varepsilon}{2} \left[ \frac{2}{2q \log (2q)} + \log \frac{\log(2p)}{\log(2q)} \right] \qquad \ldots \ (7\cdot 1)$$

Now if $p \leqslant kq^t$, $\log 2p \leqslant \log k/2^{t-1} + t \log (2q)$, $k$ being finite and $\log 2p/\log(2q) \leqslant t + (\log k')/\log (2q)$ when $k' = k/2^{t-1}$

$$\leqslant t + \varepsilon_1$$

$\therefore \quad \log \dfrac{\log (2p)}{\log (2q)} \leqslant \log (t + \varepsilon_1)$ as $p$ and $q \to \infty$ where $\varepsilon_1$ is as small as we like and is positive. It follows from (7·1) that $T = \lim T_{pq} = 0$.

**8.** *Application to the hypergeometric Series.*

$$1 - \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha \cdot \beta(\alpha+1)(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} + \cdots\cdots$$

The series is semi-convergent if $0 \leqslant \alpha + \beta - \gamma < 1$.

$$\sum{}'_{p+q} = \sum{}_{2q} + \frac{\alpha(\alpha+1)\cdots(\alpha+2q-1)\beta(\beta+1)\cdots(\beta+2q-1)}{1\cdot 2\cdots 2q\cdot\gamma(\gamma+1)\cdots(\gamma+2q-1)} + \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+2p-3)\cdot\beta(\beta+1)\cdots(\beta+2p-3)}{1\cdot 2\cdots(2p-2)\,\gamma(\gamma+1)\cdots(\gamma+2p-3)}.$$

But $\dfrac{\alpha(\alpha+1)\cdots(\alpha+2q-1)}{1\cdot 2\cdots 2q} = \dfrac{(2q)^{\alpha-1}}{\Gamma(\alpha)}(1+\varepsilon)$ where $\varepsilon \to 0$ as $q \to \infty$.

$$\therefore \quad \frac{\alpha(\alpha+1)\cdots(\alpha+2q-1)\cdot\beta(\beta+1)\cdots(\beta+2q-1)}{1\cdot 2\cdots 2q\cdot\gamma(\gamma+1)\cdots(\gamma+2q-1)}$$

$$= \frac{(2q)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{(2q)^{\beta-1}}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{(2q)^{\gamma-1}} \; \frac{(1+\varepsilon)(1+\varepsilon')}{(1+\varepsilon'')}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} (2q)^{\alpha+\beta-\gamma-1}(1+\eta_{2q}) \text{ where } \eta_{2q} \to 0 \text{ as } q \to \infty.$$

$$\therefore \quad \sum{}'_{p+q} = \sum{}_{2q} + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \left[(2q)^{t-1}(1+\eta_{2q}) + (2q+2)^{t-1}(1+\eta_{2q+2}) + \cdots + (2p-2)^{t-1}(1+\eta_{2p-2})\right] \text{ where } t = \alpha+\beta-\gamma.$$

$$= \sum{}_{2q} + \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \left[\{(2q)^{t-1} + (2q+2)^{t-1} + \cdots (2p-2)^{t-1}\} + \eta\{(2q)^{t-1} + \cdots + (2p-2)^{t-1}\}\right]$$

where $\eta$ lies between the greatest and least of $\eta_{2q}, \eta_{2q+2}, \ldots, \eta_{2p-2}$ and hence $\eta \to 0$ as $p$ and $q \to \infty$.

Now $0 \leqslant t < 1$ and hence $0 < 1-t \leqslant 1$. Let $1-t=f$. Then

$$1/(2q)^f + 1/(2q+2)^f + \cdots + 1/(2p-2)^f$$

$$= \frac{1}{2^f}\left[1/q^f + 1/(q+1)^f + \cdots + 1/(p-1)^f\right]$$

$$= \frac{1}{2^f} R_{pq} \text{ (say).}$$

We have $\int_q^p \frac{dx}{x^f} < R_{pq} < \frac{1}{q^f} + \int_q^p \frac{dx}{x^f} - \int_{p-1}^p \frac{dx}{x^f}$.

But since $f > 0$, $1/q^f \to 0$ and $\int_{p-1}^{p} dx/x^f \to 0$ as both $p$ and $q \to \infty$;

and hence

$$R_{p,q} \sim \int_q^p dx/x^t = \frac{p^t - q^t}{t} \text{ if } t \neq o$$

and $R_{pq} \sim \log p/q$ if $t=0$.

$$\therefore \quad \Sigma_{p+q} = \Sigma_{2q} + \frac{1}{2^f} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1+\eta)\left[\frac{p^t - q^t}{t}\right](1+\eta') \quad \ldots \quad (8{\cdot}1)$$

where $\eta$ and $\eta' \to 0$ as $p$ and $q \to \infty$, $t \neq 0$.

We may consider the following cases now :—

*Case I*: $t=0$. In this case we have by (8·2)

$$\Sigma'_{p+q} = \frac{\Gamma(\gamma)}{2\,\Gamma(\alpha)\Gamma(\beta)}\left(\log \frac{p}{q} + \eta''\right) \text{ where } \eta'' \to 0 \text{ as } p \text{ and } q \to \infty.$$

This last expression tends to zero with $\log p/q$.

i.e. if $p/q \to 1$, i.e. if $(p-q)/q \to 0$.

If T be the change desired in the sum of the series

$$T = \frac{\Gamma(\gamma)}{2\Gamma(\alpha)\Gamma(\beta)} \lim \log \frac{p}{q} \quad \text{i.e.} \quad \frac{2\,T\cdot\Gamma(\alpha).\Gamma(\beta)}{\Gamma(\gamma)} = \operatorname{Lim} \frac{p}{q} \quad \ldots \quad (8{\cdot}2)$$

so that in this case $p = O(q)$.

*Case II*: $t \neq 0$

If $p/q \to 1+h$ where $h$ is positive, then $(p^t - q^t)/t$ becomes infinite with $p$ and $q$.

Hence it follows from (8·1) that $\Sigma_{p+q}$ diverges.

But if $p/q \to 1$ and $p/q = 1 + \varepsilon_1$ where $\varepsilon_1$ is arbitrarily small, then $p^t - q^t = q^t\ [(1+\varepsilon_1)^t - 1] = q^t\ [\varepsilon_1 t + k\varepsilon_1^2]$ where $k$ is finite and hence

$$(p^t - q^t)/t = \varepsilon_1 q^t + k\varepsilon_1^2 q^t$$

Suppose further that $\varepsilon_1 q^t \to g$ as $q \to \infty$, $g$ being finite $k\varepsilon_1^2 q^t$ must necessarily tend to zero. In this case $\log \varepsilon_1 + t \log q \to \log g$.

$$\therefore \qquad \log \{(p-q)/q\} + t \log q \to \log g.$$

$$\text{and} \qquad (p-q)q^{t-1} = (p-q)/q^f \to g$$

so that $p - q = O(q^f)$

The change in the sum of the series, in this case, will be $\frac{g}{2^f}\Gamma(\gamma)/\Gamma(\alpha)\Gamma(\beta)$ if $(p-q)/q^f \to g$ as $q \to \infty$; for example if

$$p - q = \text{the integer nearest to } g\ q^f.$$

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## GLEANINGS

### Fisher's Combination-Puzzle

53. Sixteen passengers on a liner discover that they are an exceptionally representative body. Four are Englishmen, four are Scots, four are Irish and four are Welsh. There are also four, each of four different ages 35, 45, 55 and 65 and no two of the same age are of the same nationality. By profession also, four are lawyers, four soldiers, four doctors and four clergymen, and no two of the same profession are of the same age or of the same nationality.

It appears, also, that four are bachelors, four married, four widowed, and four divorced, and that no two of the same marital status are of the same profession, or the same age or the same nationality. Finally, four are Conservatives, four Liberals, four Socialists, and four Fascists and no two of the same political sympathies are of the same marital status, or the same profession, or the same age or the same nationality. Three of the Fascists are known to be an unmarried English lawyer of 65, a married Scotch soldier of 55, and a widowed Irish doctor of 45. It is then easy to specify the remaining Fascist.

It is further given that the Irish socialist is 35, the Conservative of 45 is a Scotchman, and the Englishman of 55 is a clergyman. What do you know of the Welsh Lawyer? (p. 94 'The Design of Experiments' by R. A. Fisher.)

# NOTES AND DISCUSSIONS

## A relation between direction-ratios and direction-cosines of a straight line

We know that in a system of rectangular co-ordinates the direction-ratios $l$, $m$, $n$ and direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of any straight line in space satisfy the relation

$$\sum l^2 + \sum \cos^2\alpha = 2. \qquad \ldots \quad (1)$$

It would be interesting to know conversely what (1) would imply about the nature, viz. rectangular or otherwise, of the system of co-ordinates. A result of this kind is given in theorem A below and theorem B is a kind of converse.

THEOREM A: *If in a real co-ordinate system a real straight line exists whose direction-ratios $l$, $m$, $n$ and direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ satisfy the relation $\sum l^2 + \sum \cos^2\alpha = 2$, then $l$, $m$, $n$ are respectively equal to $\cos\alpha$, $\cos\beta$, $\cos\gamma$; and all three, two, or one of the angles between the axes are right angles according as the straight line is parallel to none of the axes and none of the co-ordinate planes, parallel to one of the axes, or parallel to none of the axes but parallel to one of the co-ordinate planes.*

Proof :—From the relations

$$\cos\alpha = l + m\cos\nu + n\cos\mu \qquad \ldots \quad (2)$$

and two similar relations we have

$$2 = \sum l^2 + \sum\cos^2\alpha = \sum l^2\,(2 + \cos^2\mu + \cos^2\nu) + 2\sum mn\,(\cos\mu\cos\nu + 2\cos\lambda).$$

Hence

$$\sum l^2\,(\cos^2\mu + \cos^2\nu) + 2\sum mn\cos\mu\cos\nu = 0,$$

since $\quad \sum l^2 + 2\sum mn\cos\lambda = 1.$

That is $\sum (m\cos\nu + n\cos\mu)^2 = 0.$

Now, since the system of co-ordinates and the line are both real, we have

$$m\cos\nu + n\cos\mu = 0, \text{ etc.} \qquad \ldots \quad (3)$$

From (2) and (3) we obtain $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

Multiplying equations (3) by $l$, $m$, $n$ respectively and adding we obtain

$$\sum mn\cos\lambda = 0. \qquad \ldots \quad (4)$$

From (3) and (4) we have

$$mn\cos\lambda = 0, \quad nl\cos\mu = 0, \quad lm\cos\nu = 0. \qquad \ldots \quad (5)$$

I. If the line is parallel to none of the axes and none of the co-ordinate planes, then $l\,m\,n \neq 0$, and from (5) it follows that $\cos\lambda = 0$, $\cos\mu = 0$, $\cos\nu = 0$.

II. If the line is parallel to one of the axes, say OX, $l = 1$, $m = 0$, $n = 0$ and from (3) it follows that $\cos\mu = 0 = \cos\nu$.

III. If the line is parallel to one of the co-ordinate planes, YOZ, and none of the axes, then $l = 0$, $m\,n \neq 0$, and from (3) or (5) it follows that $\cos\lambda = 0$.

I, II, and III prove the theorem.

THEOREM B: *In a real co-ordinate system in which at least one of the angles between the axes is a right angle, there exists at least one straight line whose direction-ratios and direction-cosines satisfy the relation* $\sum l^2 + \sum \cos^2\alpha = 2$.

Proof: I′. All the angles are right angles.

In this case $\sum \cos^2\alpha = 1 = \sum l^2$.

II′. Let two of the angles be right say XOY, XOZ.

Any line parallel to OX has direction-ratios 1, 0, 0 and direction-cosines 1, 0, 0. Hence the result.

III′. Let one of the angles, say YOZ, be a right angle.

The line of intersection of the plane YOZ and the plane normal to OX has direction ratios 0, $m$, $n$, and direction-cosines 0, $\cos\beta$, $\cos\gamma$ such that $m^2 + n^2 = 1 = \cos^2\beta + \cos^2\gamma$. I′, II′, III′ prove the theorem.

Remark: In any real co-ordinate system and for every real straight line, $\sum l^2 + \sum \cos^2\alpha = 2 + \sum (m\cos\nu + n\cos\mu)^2 \geqslant 2$. Hence our hypothesis of theorem A is that $\sum l^2 + \sum \cos^2\alpha =$ the minimum value of the function.

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## A Generalisation of the Radix Theorem

If $g_1, g_2, g_3, \ldots$ be an infinite sequence of positive integers and $G(0)=1$, $G(r)=g_1\, g_2\, g_3 \ldots g_r$ for $r > 0$, then any number $n$ can be expressed *uniquely* in the form:

$$n = \sum_{r=0}^{k} c_r\, G(r), \text{ where } 0 \leqslant c_r < g_{r+1} \quad (0 \leqslant r \leqslant k). \qquad \ldots \quad (1)$$

Proof: For if $x \neq 1$, we have

$$\frac{1 - x^{G(r+1)}}{1 - x^{G(r)}} = \frac{1 - x^{G(r) g_{r+1}}}{1 - x^{G(r)}}$$

$$= 1 + x^{G(r)} + x^{2G(r)} + \cdots + x^{(g_{r+1} - 1)G(r)}.$$

Now if $\quad 0 < x < 1$,

$$\frac{1}{1-x} = \prod_{r=0}^{\infty} \frac{1 - x^{G(r+1)}}{1 - x^{G(r)}} = \prod_{r=0}^{\infty} \left(1 + x^{G(r)} + \cdots + x^{(g_{r+1}-1)\, G(r)}\right)$$

$$= S_0 + S_1\, x + S_2\, x^2 + \cdots + S_n\, x^n + \cdots$$

where $S_n$ represents the number of solutions of (1).

But $\quad \dfrac{1}{1-x} = 1 + x + x^2 + \cdots\cdots$

Comparing coefficients we have

$S_0 = S_1 = S_2 = \cdots = S_n = \cdots = 1$ and hence the assertion.

If in particular, $g_1 = g_2 = \cdots = g$ we get that the representation $n = c_0 + c_1\, g + c_2\, g^2 + \cdots$ is always possible and unique which is the well-known Radix Theorem.

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## A Note on the Normals to a Conic

It is well known that every central conic has four foci and that every conic through the four foci is a rectangular hyperbola. We seek to define these rectangular hyperbolas as conic loci associated with the normals to the given conic $\Sigma$.

THEOREM: The locus of a point P which moves such that the sum of the angles made by the four normals from P to any central conic $\Sigma$, with a fixed line is constant, is a rectangular hyperbola through the four foci of $\Sigma$.

If $P(x', y')$ be any such point, then the condition that the line

$$y - y' = m(x - x') \qquad \dots \quad (1)$$

should be a normal to the conic

$$\Sigma: \qquad ax^2 + by^2 = 1 \qquad \dots \quad (2)$$

is

$$m^4 a x'^2 - 2m^3 a x' y' + m^2\{bx'^2 + ay'^2 - (a-b)^2/ab\} - 2mbx'y' + by'^2 = 0 \quad (3)$$

The equation (3) being a quartic in $m$ gives four values for $m$ which are the tangents of the angles made by the four normals from P with the axis of $x$. If these angles be $\theta_r (r = 1, 2, 3, 4)$ we get easily that

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{2x' y' ab}{ab(x'^2 - y'^2) + (a-b)} \qquad \dots \quad (4)$$

From (4) we see that if $P(x' y')$ moves so that $\sum\theta_r$ is a constant, the locus of P is the conic

$$R_\lambda: \qquad \lambda\{ab(x^2 - y^2) + (a-b)\} - 2ab\,xy = 0 \qquad \dots \quad (5)$$

(where $\lambda = \tan \sum\theta_r$) which is easily seen to be a rectangular hyperbola through the four foci of $\Sigma$.

Now, if $\sum\theta_r$ vanishes the corresponding conic $R_o$ degenerates into the two principal axes of $\Sigma$, while if $\sum\theta_r$ is an odd multiple of $\pi/2$, the corresponding conic $R_\infty$ is the locus of foci of all conics inscribed in the rectangle formed by the tangents to $\Sigma$ at the extremities of the principal axes.

If the conic $\Sigma$ is a parabola there is only one finite focus F. Taking the equation of the parabola in the standard form

$$\Sigma: \qquad y^2 = 4ax \qquad \dots \quad (6)$$

the condition that the line (3) should be a normal is

$$am^4 + m^2(2a - x') + my' = 0. \qquad \dots \quad (7)$$

From (7) we infer that one of the four normals is always parallel to the axis of the parabola while the sum of the angles made by the other three normals with axis is given by

$$\tan(\theta_1 + \theta_2 + \theta_3) = y'/(x' - a) \qquad \dots \quad (8)$$

If, now, $\sum\theta$ is a constant and $\tan\sum\theta = \lambda$ we see that the locus of P is the straight line

$$y' = \lambda(x' - a) \qquad \dots \quad (9)$$

Evidently for different values of $\lambda$, the equation (9) represents the pencil of straight lines through the focus F.

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## The foci of the Steiner Circum-conic of a triangle

On p. 203, Vol. IV of this Journal, Mr. N. Durairajan identifies the two points which satisfy Morleys' problem :

> 'To draw from the vertices of a triangle to the opposite sides three segments of the same length $\lambda$, all passing through one point'

as the foci of the Steiner circum-conic of the triangle. The Morleys have apparently intended their solution as a 'pleasant reverberation in the investigation of elementary questions, of the ideas of Riemann, Poincaré, Klein, and others' and believe that 'the tradition of handling them as Euclid or Descartes might have handled them is very hampering.'* We submit below a solution as Euclid may have handled it and leave the reader to judge the extent of the 'hampering.'

Let S be the required point and ABC the given triangle and AS, BS, CS meet the sides BC, CA, AB in D, E, F respectively. Draw through the vertices, straight lines parallel to the opposite sides so as to form the triangle A′B′C′, where AA′, BB′, CC′ meet in the common centroid of the two triangles.

Now $\triangle$ AC′D = $\triangle$ ABC′ in area (BD $\parallel$ AC′)

= $\triangle$ BC′E in area (BC′ $\parallel$ AE)

If AD = BE, the altitudes of the equivalent triangles AC′D, BC′E perpendicular to the equal bases are equal.

Hence C′S bisects $A\hat{S}B$.

Similarly, if BE = CF, A′S bisects $B\hat{S}C$ and B′S bisects $A\hat{S}C$. Hence from the well-known property that two tangents to a conic subtend equal or supplementary angles at a focus, we can identify S as a focus of the Steiner inscribed ellipse of the triangle A′B′C′, which touches the sides at their mid-points.

The problem is thus completely solved with the help of two simple theorems in elementary geometry and a focal property of the conic.

There are two solutions corresponding to the two foci of the conic, which may also be considered as the Steiner circum-conic of ABC, whose tangents at the vertices are parallel to the opposite sides.

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* Vide p. vi, Preface, *Inversive Geometry* by F. Morley and F. V. Morley.

If S, S′ are the two foci, AS, AS′ are equally inclined to the tangent at A and therefore also to BC. Hence if AS′ meets BC in D′, AD = AD′.†

To evaluate AD′, we make use of another elementary result, viz.

$$\frac{AS}{AD}+\frac{BS}{BE}+\frac{CS}{CF}=2$$

$$\therefore \qquad AS+BS+CS=2\ AD$$

Similarly, $\qquad AS'+BS'+CS'=2\ AD'=2\ AD$

$$\therefore \qquad 4\ AD=(AS+AS')+(BS+BS')+(CS+CS')$$

= three times the major axis of the Steiner conic

i.e. AD = three-fourths of the major axis,
a result noted by Mr. N. Durairajan without proof.

*Mysore,*
8—7—1937.

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## A Proof of Fermat's and Wilson's Theorems

**1.** The object of this paper is to discuss some properties of periodic decimals and apply these to prove some of the known theorems in the theory of numbers. In what follows, all symbols represent positive integers; $p$ and $q$ are used to denote different odd primes; $s$ denotes the scale of notation. We denote by $\{1/p, s,\}$ the fraction $1/p$ converted into a decimal in the scale of notation $s$. Let $a_1, a_2, \cdots$ denote the digits and $r_0=1$, $r_1$, $r_2$, $\cdots$ the least positive remainders, modulo $p$, obtained during the process of conversion. We shall speak of $r_0, r_1, r_2, \cdots$ merely as remainders, for shortness. We have the following fundamental relations:

$$\begin{aligned} sr_0 &= s = a_1 p + r_1 \\ sr_1 &= a_2 p + r_2 \\ \cdots & \quad \cdots \quad \cdots \\ sr_i &= a_i p + r_{i+1} \qquad 0 \leqslant r_i < p \qquad \ldots \quad (1) \end{aligned}$$

and

$$\left.\begin{aligned} r_i &\equiv s^i \ (\mathrm{mod}\cdot p) \\ s^\lambda r_i &\equiv s^{i+\lambda} \ (\mathrm{mod}\cdot p) \end{aligned}\right\} \qquad \ldots \quad (2)$$

while

$$\{1/p.\ s\} = \cdot a_1 a_2 \cdots\cdots = a_1/s + a_2/s^2 + a_3/s^3 + \cdots\cdots \qquad \ldots \quad (3)$$

† Vide Ex. 22, p. 217, ibid.

**2.** The following results * are immediate consequences of (1) and (2):

THEOREM I. $\{1/p, s\}$ is a purely recurring decimal, the number† of digits $k$ in a period being a divisor of $p-1$; if $r$ is prime to $p$, the number of digits when $r/p$ is converted into a decimal in the scale $s$ is also $k$.

*Proof.* That $\{1/p, s\}$ is purely recurring follows from the fact that all remainders $r_i$ are different and can take only $p-1$ different values. Also, the remainders when $a/p$, $b/p$, $(a, p)=1$, $(b, p)=1$, are converted into decimals are different though they are the same in number provided $b$ does not occur in among the remainders of $a/p$. Also the total number of remainders when all the fractions $1/p$, $2/p$. $\cdots (p-1)/p$ are converted into decimals is exactly $p-1$. Hence $k$ divides $p-1$.

THEOREM II. If the number of digits $k$ in a period is even and equal to $2m$, then $r_m = p-1$.

*Proof.* We have, by (2)

$$r_m \equiv s^m \pmod{p}$$

$$s^m r_m \equiv r_{2m} \equiv r_0 \equiv 1 \pmod{p}$$

Hence $\quad r_m{}^2 \equiv 1 \pmod{p}$, that is $(r_m - 1)(r_m + 1) \equiv 0 \pmod{p}$.

Now $0 \leqslant r_m < p$. Hence $r_m + 1$ must be divisible by $p$ so that $r_m = p - 1$.

**3.** *Proof of Fermat's theorem:* Let the fraction $1/p$ be converted into a decimal in the scale $s$ where $(s, p)=1$. The number of digits $k$ in a period is a divisor of $p-1$. Hence $\{1/p, s\}$ can be regarded as a fraction of period $p-1$ which when converted into vulgar fraction must have $(s-1, s-1, \cdots s-1 \cdots p-1$ times) for denominator. This denominator must be divisible by $p$. But the denominator is

$$(s-1)s^{p-2} + \cdots + (s-1) \cdot 1 = s^{p-1} - 1$$

Hence $s^{p-1} - 1 \equiv 0 \pmod{p}$ which is Fermat's theorem.

**4.** *Proof of Wilson's theorem*: Let $s$ be a primitive root of $p$, that is

$$s^h \equiv 1 \pmod{p}$$

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* Cf. Bachmann, "*Niedere Zahlentheorie*", 1st part, 351—363.

† If $p$ is not a prime, $k$ is a divisor of $\phi(p)$, the number of members less than $p$ and prime to it.

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is true for $h = p-1$ but not for any $h < p-1$. In this case, the period of $\{s/p\}$ is actually $p-1$ and the remainders $r_0, r_1, \cdots r_{p-1}$ will contain all the numbers $1, 2, \cdots p-1$ in some order. Therefore by (2),

$$s \times s^2 \times \cdots \times s^{p-1} \equiv r_0 \cdot r_1 \cdots r_{p-1} \equiv (p-1)! \ (\text{mod. } p) \cdots \quad (3)$$

But $p-1$ is even and so by Theorem II, $r_{(p-1)/2} - p - 1$ so that $s^{(p-1)/2} \equiv p - 1 \equiv -1$ (mod. $p$). So (3) gives

$$(p-1)! \equiv s^{p(p-1)/2} \equiv (-1)^p \equiv -1 (\text{mod. } p)$$

which is Wilson's theorem.

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## On the construction of a magic hypercube of odd order

§ 1. Let the number $a_p + b_q + c_r + d_s$ be set at the point $(p, q, r, s)$ in the fourth dimensional space of axes OX, OY, OZ, OU for all integer values of $p, q, r, s$. Let $a_{p+m} = a_p$, $b_{q+m} = b_q$, $c_{r+m} = c_r$, $d_{s+m} = d_s$ for every value of $p, q, r, s$ and a particular integer value of $m$. When this condition is satisfied, $a_{p-m} = a_p$.

When the above conditions are satisfied, the sum of $m$ consecutive numbers along a line parallel to any one of the lines $x = \pm y = \pm z = \pm u$, is constant. For, the sum of $m$ figures beginning with $a_p + b_q + c_r + d_s$ is

$$\sum_{n=0}^{m-1} \left( a_{p+n} + b_{q \pm n} + c_{r \pm n} + d_{s \pm n} \right)$$

$$= (a_1 + \cdots + a_m) + (b_1 + \cdots + b_m) + (c_1 + \cdots + c_m) + (d_1 + \cdots + d_m)$$

$$= \text{constant.}$$

Hence the hyperparallelopiped whose four adjacent edges are four of the eight lines parallel to $x = \pm y = \pm z = \pm u$ starting from $a_p + b_q + c_r + d_s$ and containing $m$ elements along each of the edges is a magic hyperparallelopiped which can be arranged as a Magic Hybercube.

2. If $m$ is even, every element in the hypercube will be repeated, and if $m$ is odd no element will be repeated.

For consider the general term of the hypercube whose four adjacent edges are parallel to say (i) $x = y = z = u$, (ii) $x = y = -z = -u$, (iii) $x = -y = z = -u$, (iv) $x = -y = -z = u$. Then the general term is

$$a_{p+n_1+n_2+n_3+n_4} + b_{q+n_1+n_2-n_3-n_4} + c_{r+n_1-n_2+n_3-n_4} + d_{s+n_1-n_2-n_3+n_4} ;$$

where $n_1, n_2, n_3, n_4$ take the values $0, 1, 2, \ldots m-1$.

If this term is to be the same as $a_p + b_q + c_r + d_s$, then

$$n_1 + n_2 + n_3 + n_4 = \alpha m \quad \ldots \quad (1)$$
$$n_1 + n_2 - n_3 - n_4 = \beta m \quad \ldots \quad (2)$$
$$n_1 - n_2 + n_3 - n_4 = \gamma m \quad \ldots \quad (3)$$
$$n_1 - n_2 - n_3 + n_4 = \delta m \quad \ldots \quad (4)$$

where $\alpha$, $\beta$, $\gamma$, $\delta$, are integers.

Adding (1), (2), (3), and (4),

$$n_1 = (\alpha + \beta + \gamma + \delta)m/4$$

Adding (1) and (2) and subtracting (3) and (4)

$$n_2 = (\alpha + \beta - \gamma - \delta)m/4$$

Similarly we get

$$n_3 = (\alpha - \beta + \gamma - \delta)m/4$$
$$n_4 = (\alpha - \beta - \gamma + \delta)m/4$$

Since $n_1$, $n_2$, $n_3$, $n_4$ are positive integers not greater than $m-1$, $m$ should be divisible by 2 or 4 in order that the $n$'s may have integer solutions. Hence when $m$ is an odd integer no element would be repeated. When $m$ is even the elements would be repeated.

It may be shown that when $m$ is divisible by 2 but not by 4 each element is repeated 8 times and that when $m$ is divisible by 4 each element is repeated 16 times.

**3.** *Construction of a magic hypercube out of the first 81 natural numbers.*

Let $m = 3$, and the elements be the numbers from 1 to 81.

Let $d_1 = 1,\ d_2 = 2,\ d_3 = 3$
$c_1 = 0,\ c_2 = 3,\ c_3 = 6$
$b_1 = 0,\ b_2 = 9,\ b_3 = 18$
$a_1 = 0,\ a_2 = 27,\ a_3 = 54.$

The rows in the squares of the magic hypercube are parallel to $x = -y = z = -u$.

The rows of the squares are parallel to $x = -y = -z = u$.

The columns in the squares are parallel to $x = y = z = u$.

The columns of the squares are parallel to $x = y = -z = -u$.

Table II gives the Magic Hypercube. The properties are :—

(1) The sum of the three figures in any row or column of any square of the hypercube is 123.

(2) The sum of any set of the three corresponding figures taken from the three squares in any row or column of the Hypercube is also 123.

Table I gives the suffixes of the several elements.

TABLE I

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1111 | 2323 | 3232 | 2332 | 3211 | 1123 | 3223 | 1132 | 2311 |
| 2222 | 3131 | 1313 | 3113 | 1322 | 2231 | 1331 | 2213 | 3122 |
| 3333 | 1212 | 2121 | 1221 | 2133 | 3312 | 2112 | 3321 | 1233 |
| 2233 | 3112 | 1321 | 3121 | 1333 | 2212 | 1312 | 2221 | 3133 |
| 3311 | 1223 | 2132 | 1232 | 2111 | 3323 | 2123 | 3332 | 1211 |
| 1122 | 2331 | 3213 | 2313 | 3222 | 1131 | 3231 | 1113 | 2322 |
| 3322 | 1231 | 2113 | 1213 | 2122 | 3331 | 2131 | 3313 | 1222 |
| 1133 | 2312 | 3221 | 2321 | 3233 | 1112 | 3212 | 1121 | 2333 |
| 2211 | 3123 | 1332 | 3132 | 1311 | 2223 | 1323 | 2232 | 3111 |

TABLE II

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 51 | 71 | 53 | 64 | 6 | 69 | 8 | 46 |
| 41 | 61 | 21 | 57 | 23 | 43 | 25 | 39 | 59 |
| 81 | 11 | 31 | 13 | 36 | 74 | 29 | 76 | 18 |
| 45 | 56 | 22 | 58 | 27 | 38 | 20 | 40 | 63 |
| 73 | 15 | 35 | 17 | 28 | 78 | 33 | 80 | 10 |
| 5 | 52 | 66 | 48 | 68 | 7 | 70 | 3 | 50 |
| 77 | 16 | 30 | 12 | 32 | 79 | 34 | 75 | 14 |
| 9 | 47 | 67 | 49 | 72 | 2 | 65 | 4 | 54 |
| 37 | 60 | 26 | 62 | 19 | 42 | 24 | 44 | 55 |

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## The Pedal Conic

§ **1.** The problem is to find the locus of a point which is such that the feet of the perpendiculars from it on six given lines lie on conic. This is analogous to the pedal or Simson line of a triangle.

*Notation* Let

$$L_r = x\cos\alpha_r + y\sin\alpha_r - p_r = 0 \qquad (r = 1, 2 \cdots 6)$$

be the equations of the six lines. Let the point P be $(x, y)$. The co-ordinates of the feet of the perpendiculars on the six lines are of the form $(x+X_r,\ y+Y_r)$ where

$$X_r = q_r\cos\alpha_r, \qquad Y_r = q_r\sin\alpha_r,$$

and

$$q_r = p_r - x\cos\alpha_r - y\sin\alpha_r$$

The six points $(x+X_r,\ y+Y_r)$ lie on a conic if the determinant of six rows, i.e.

$$|(x+X_r)^2,\ (x+X_r)(y+Y_r),\ (y+Y_r)^2,\ x+X_r,\ y+Y_r,\ 1| = 0$$

This is the same as

$$|X_r^2,\ X_rY_r,\ Y_r^2,\ X_r,\ Y_r,\ 1| = 0.$$

Expanding this in the form of a sum of products of complementary minors, we get

$$0 = \sum \begin{vmatrix} X_1^2 & X_1Y_1 & Y_1^2 \\ X_2^2 & X_2Y_2 & Y_2^2 \\ X_3^2 & X_3Y_3 & Y_3^2 \end{vmatrix} \times \begin{vmatrix} X_4 & Y_4 & 1 \\ X_5 & Y_5 & 1 \\ X_6 & Y_6 & 1 \end{vmatrix}$$

$$= \sum(-) \begin{vmatrix} X_2 & Y_2 \\ X_3 & Y_3 \end{vmatrix} \begin{vmatrix} X_3 & Y_3 \\ X_1 & Y_1 \end{vmatrix} \begin{vmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{vmatrix} \times |X_4\ Y_5\ 1|$$

$$\therefore\ 0 = \sum \left\{ \begin{matrix} q_1^2\, q_2^2\, q_3^2 \cdot \sin(\alpha_2-\alpha_3)\sin(\alpha_3-\alpha_1)\sin(\alpha_1-\alpha_2) \\ \times [q_4\, q_5 \sin(\alpha_5-\alpha_4) + q_5\, q_6 \sin(\alpha_6-\alpha_5) + q_6\, q_4 \sin(\alpha_4-\alpha_6)] \end{matrix} \right\}.$$

∴ The locus is a curve $S_8$ of the eighth degree. As the expression within the square brackets represents the circle circumscribing the triangle formed by $L_4\ L_5\ L_6$, the locus passes through the circular points at infinity. It is also seen that the locus passes through the 15 points of intersections of the 6 lines, through the 20 points of intersections of the 10 pairs of circles $(L_1\ L_2\ L_3)$ and $(L_4\ L_5\ L_6)$, and also the 15 Miquel points of the 15 sets of any 4 lines chosen out of the six. It also passes through the focus M of the unique biparabola, (i.e. curve of class three which has the line at infinity as a bitangent) touching the six lines. The first positive pedal of this curve w.r.t. M is a conic passing through M.

**2.** To find the locus of a point which is such that the feet of the perpendiculars on four given lines is a circle.

The locus is $|X_1^2+Y_1^2,\ X,\ Y,\ 1|=0$, that is,

$$\sum_1^4 q_1^2\ [q_2\ q_3 \sin(\alpha_2-\alpha_3)+\ \cdot\ \cdot\ +\ \cdot\ \cdot\ ]=0$$

This is apparently a quartic curve; but as the terms of the fourth degree disappear, the locus is a cubic*. It is the locus of foci of conics touching four given lines and is a circular cubic passing through the six intersections of the given lines and the Miquel point.

**3.** The locus in § 1 perhaps reduces itself to one of degree less than 8. It is seen from Salmon's Conics that the expression

$$\sum_1^4 q_1^2 \sin(\alpha_2-\alpha_3)\sin(\alpha_3-\alpha_4)\sin(\alpha_4-\alpha_2)=0$$

(which is the locus of the centre of a conic touching four lines $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4$) reduces to one of the first degree.

It should be interesting to find out how $S_8$ splits up when the six lines $L_r$ have special position, e.g. when the six lines are the mutual joins of four points.

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## ASTRONOMICAL NOTES.

### Planet notes for September, October and November 1937.

*Sun.*—During the period from the 1st September to 31st of November, the Sun will be moving in an easterly direction from 10 h. 39 m. R.A. to 16 h. 22 m. R.A. and its declination will be gradually decreasing 8° 31′ North to 0° 0′ on September 23, and then it gradually increases to 21° 33′ South at the end of November. On 23rd September, the Sun is at the Autumnal equinox and hence the duration of night and day will be the same. The effect of the Southward motion of the Sun in lessening the length of the days will be more noticeable from the 2nd half of October. The Sun's path for this period will start from the centre of Leo, and passing Virgo and Libra, will end in the middle of Scorpio at the end of November.

* SALMON, *Treatise on Con'c S ctions* 1904 P. 275. Ex. 15.
* SALMON, *Treatise on Conic Sections* 1904 P. 216. Ex. 8.

The Sun's position at the beginning of each month will be as follows:—

| | 1st Sep. | 1st Oct. | 1st Nov. |
|---|---|---|---|
| Right Ascension | 10 h. 40 m. | 12 h. 28 m. | 14 h. 24 m. |
| Declination | 8° 25′ N. | 3° 3′ S. | 14° 19′ S. |

*Moon.*—The following phases of the Moon will occur during the months of September, October and November.

| | SEPTEMBER I. S. T. | OCTOBER I. S. T. | NOVEMBER I. S. T. |
|---|---|---|---|
| New Moon | 5th at 4-24 A. M. | 4th at 5-28 P. M. | 3rd at 9-46 A. M. |
| First Quarter | 13th at 2-27 A. M. | 12th at 9-17 P. M. | 11th at 3-3 P. M. |
| Full Moon | 20th at 5-2 P. M. | 20th at 3-18 A. M. | 18th at 1-40 P. M. |
| Last Quarter | 27th at 11-13 A. M. | 26th at 6-56 P. M. | 25th at 5-34 A. M. |

The Moon will be nearest the earth (perigee) on September 24th, October 21st and November 19th and will be farthest from the earth (apogee) on 12th September, 9th October and 6th November. On 18th November there will be a partial lunar eclipse which is invisible here.

The Moon's position at the beginning of each month will be as follows:—

| | 1st Sep. | 1st Oct. | 1st Nov. |
|---|---|---|---|
| Right Ascension | 7 h. 31 m. | 9 h. 58 m. | 12 h. 58 m. |
| Declination | 18° 29′ N. | 7° 3′ N. | 10° 26′ S. |

*Mercury.*—During the early part of September, Mercury will be in retrograde motion and will be in inferior conjunction with the Sun on 14th September. On 22nd it will be stationary. From this date its motion will be eastwards like the Sun. From 12th September to 4th October it will be in Leo, and passing Virgo (5th October—20th October), Libra (21st Oct.-8th November), Scorpio (9th Nov.-27th Nov.) it enters Sagittarius at the end of November. Mercury will be at greatest elongation West on 30th September and will be at superior conjunction with the Sun on 29th October. During this period, the planet will be in conjunction with the Moon on 6th September and 3rd October and also in conjunction with Neptune on 30th September. At the beginning of September it will be so near the Sun as to be hardly visible, but by the end of the month it will be visible as a morning object till the middle of October. Again it approaches the Sun and will not be visible during the month of November.

Its position at the beginning of each month will be :—

| | September. | October. | November. |
|---|---|---|---|
| Right Ascension | 11 h. 52 m. | 11 h. 24 m. | 14 h. 31 m. |
| Declination | 3° 36′ S. | 5° 20′ N. | 14° 47′ S. |

*Venus.*—During the months of September, October and November Venus will be visible as a morning object rising rather late day by day. It will have great brilliance. Its path for this period will start in Cancer, extend through Leo and Virgo and will be in Libra by the end of November. It will be in conjunction with the Moon on 2nd September, 2nd October and 1st November and it will be in conjunction with Regulus and Neptune on 24th September and 11th October respectively.

Its position at the beginning of each month will be as follows:—

| | September. | October. | November. |
|---|---|---|---|
| Right Ascension | 8 h. 14 m. | 10 h. 38 m. | 13 h. 0 m. |
| Declination | 19° 25′ N. | 9° 46′ N. | 4° 40′ S. |

*Mars.*—The planet Mars will be visible as an evening object during these months. Its motion is east and slightly less rapid than that of the Sun and its path is from Scorpio to Capricorn. It is in conjunction with the Moon on 12th September, 11th October and 9th Nov. It will be in conjunction with Jupiter on 29th October.

Its positions will be as follows :—

| | September. | October. | November. |
|---|---|---|---|
| Right Ascension | 16 h. 40 m. | 18 h. 2 m. | 19 h. 35 m. |
| Declination | 24° 50 m. S. | 25° 48 m. S. | 23° 38 m. S. |

*Jupiter.*—Like Mars the planet Jupiter is an evening object and can be seen till midnight during the early part of September and during the early part of the evening hours at the end of November. At the beginning of September its motion is retrograde; on 13th September it is stationary, and after that date its motion is direct. The planet will be in Sagittarius till 12th November and then it enters the constellation Capricorn. It will be in conjunction with the Moon on the following dates: 15th September, 12th October, and 9th November. The planet is a special object of interest to amateur Astronomers, because the ever-changing aspect of its Satellites affords them an interesting study :

Its positions will be as follows :—

| | September. | October. | November. |
|---|---|---|---|
| Right Ascension | 19 h. 17 m. | 19 h. 18 m. | 19 h. 31 m. |
| Declination | 22° 42′ S. | 22° 42′ S. | 22° 16′ S. |

*Saturn.*—The planet Saturn will be in Pisces throughout the period. The motion of Saturn will be retrograde and it will rise earlier from night to night. Saturn will be in conjunction with the Moon on 21st September, 18th October and 14th November and it will be in opposition with the Sun on 25th September.

Its positions will be as follows:—

| | September. | October. | November. |
|---|---|---|---|
| Right Ascension | 0 h. 17 m. | 0 h. 9 m. | 0 h. 1 m. |
| Declination | 0° 58′ S. | 1° 53 m. S. | 2° 40′ S. |

*Uranus.*—The planet will be visible in the evening through powerful telescopes. Its motion is retrograde and it will be in opposition with the Sun on 4th November.

Its position will be:—

| | September. | October. | November. |
|---|---|---|---|
| Right Ascension | 2 h. 45 m. | 2 h. 43 m. | 2 h. 38 m. |
| Declination | 15° 31′ N. | 15° 20′ N. | 14° 58′ N. |

*Neptune.*—This planet moves in an easterly direction. This is an evening object and is visible only through powerful telescopes. It will be in conjunction with the Sun on 11th September.

Its positions will be as follows:—

| | September. | October. | November. |
|---|---|---|---|
| Right Ascension | 11 h. 19 m. | 11 h. 23 m. | 11 h. 26 m. |
| Declination | 5° 35 m. N. | 5° 10′ N. | 4° 47′ N. |

*Other astronomical notes and extracts.*

In a letter to *Nature* dated 13th April from Budapest Dr. J. Barnothy and Dr. M. Forro have found by subjecting to harmonic analysis the hourly records of the intensity of Cosmic rays observed for a period of more than 2 years by means of 'counters' inclined at various angles to the horizon, that there is a straight line relation between the maximum of the first harmonic wave (sidereal time) and the total thickness of the absorbing material through which the cosmic rays have passed. They consider this effect to be related to the galactic rotation.

*Is τ Tauri a double star.* (*Jour. of the Br. Astr. Association 6—4—1937*)

Mr. Bertrand M. Peek observed on 22nd January the occultation of τ Tauri and he saw two successive flickers lasting for about 0·4 of a second. In February 1929 also he observed the star going out

in two stages. He infers that the star may be a double star of 0·2″ separation. Mr. G. F. Kellaway at Yeovil also reported having seen the same phenomenon with a separation of 0·5 second. It is worth while to observe future occultations of this star and come to a definite conclusion on this point.

*Solar eclipse of June 19, 1936. (Nature, July 3, 1937).*

This article contains an account of a joint meeting of the Royal Society and the Royal Astronomical Society on May 27 to receive the reports on the expeditions for the total eclipse of June 19. The eclipse day was cloudy at Kamishari in Hokkaido; only the 2nd half of the partial phase could be observed and Dr. Royds of the Kodaikkanal Observatory who was in the expedition obtained flashlight spectra. He has also measured the displacement of the Fraunhofer lines at the centre and limb of the Sun's disc with and without an eclipse.

*Lunar Craters.*

Dr. Spencer in an article in *Nature*, April 17, has shown that the meteoric theory of the Lunar craters is sound physically and dynamically. The absence of large explosion craters from the surface of the earth has hitherto been an objection to the meteoric theory. Dr. Spencer has shown that this is no longer a fatal difficulty and that the explosion of meteors would give rise to circular craters. This hypothesis therefore deserves serious consideration in the light of Dr. Spencer's suggestions.

*Earthquakes in India.*

This is the subject-matter of the Presidential Address by Mr. W. D. West at the meeting of the last Indian Science Congress. The whole country is divided into 3 zones, of which only one is active and this again is subdivided into the following zones (1) Cutch; (2) Baluchistan; (3) North India; (4) Assam; (5) Burma.

*Astro-Physical Journal, Vol. 85, No. 5* contains the following investigations of astronomical importance.

1. The determination of the amount of carbon-dioxide above the reflecting layer in the atmosphere of Venus by Arthur Adel.

2. Mr. S. Chandrasekhar has established several interesting theorems on the pressure in the interior of a star.

*Astro-Physical Journal, Vol. 85, No. 4* contains an article on "The Physical processes in gaseous nebulae" by D. H. Menzel.

At first a series of formulae dealing with the physical state of gaseous nebulae are derived. The total emission and absorption of radiation by atomic hydrogen are then evaluated. The equations determining the statistical equilibrium of the assembly and partition of atoms into different states are also derived.

*Astro-Physical Journal, Vol. 85, No. 3* contains an article on Radiation pressure in Galactic nebulae by J. L. Greenstein. Observation of interstellar selective absorption shows the existence of small solid particles with radii approaching the wavelength of light. But investigations by Struve and others show the presence of large particles in bright galactic nebulae. This difference is due to the enormous radiation pressure of the hot stars illuminating the nebulae.

A new binary star of large parallax (0″·053 ± 0·006) has been discovered having a proper motion of 0″·66 by G. P. Kuiper. The visual magnitude of the companion is 13·4.

*A. P. J. Vol. 85.* (2) "New identifications of solar lines" is an article by C. E. Moore establishing the presence of lines due to Li and Rb in the disc spectrum. Os and Ir lines also are present in the Solar spectrum, though only suspected till now. Tu has been detected only by C. E. Moore, occurring in an ionised state. The elements Sn, Ta, Tb and Ne are suspected.

Maharaja's College. *Trivandrum.*

H. SUBRAMANI AIYAR.

# REVIEWS

M. NICOLESCO, '*Les Fonctions Polyharmoniques*'
Actualités Scientifiques et Industrielles, No. 331. Hermann & Co., Paris (1936)
Pp. 54, price 15 Francs.

This book contains a collection of results which embody the chief properties of what are called 'Polyharmonic Functions.' The notion of Polyharmonic Functions is a natural generalisation of the concept of harmonic functions. Let $u(x_1, \ldots x_n) = u(\mathrm{M})$ be a function defined in some domain D of the space of $n$ dimensions.

Let $\Delta = \sum \partial^2 / \partial x_i^2$ be the Laplace operator and

$$\overset{s}{\Delta} = \Delta \overset{s-1}{\Delta}, \qquad (s = 1, 2, \ldots)$$

where $\overset{\circ}{\Delta} \equiv 1$. If $u(\mathrm{M})$ be such that $\Delta u(\mathrm{M}) = 0$ in a domain D, $u(\mathrm{M})$ is said to be harmonic in D. If

$$\overset{p}{\Delta} u(\mathrm{M}) = 0,$$

the function $u(\mathrm{M})$ is said to be polyharmonic of order $p$ ($p \geqslant 1$). When $p = 2$, the function is said to be biharmonic. Though these functions have appeared early enough in mathematics, they have been mainly studied only in relation to boundary problems analogous to Dirichlet's problem for harmonic functions. A systematic study of the intrinsic properties of polyharmonic functions has been undertaken only recently and the major portion of these researches is due to the author of the pamphlet.

The first chapter is devoted to the study of the various conditions in order that a function may be polyharmonic in a domain, analogous to those for harmonic functions. In the second chapter, are considered extensions and analogues of Liouville's theorem for harmonic functions—that if a function harmonic in the whole plane is bounded (or bounded on one side only), it reduces to a constant. Similar results are discussed in relation to Picard's theorem on exceptional values, and Harneck's and Montel's theorems on the convergence of sequences of harmonic functions. The third and last chapter is devoted to the study of boundary problems corresponding to Dirichlet's problem for harmonic functions. Lastly a short discussion is given of the relation of biharmonic functions to the problems arising in the equilibrium of elastic bodies under given forces. The booklet is valuable as giving in a concise form the main results so far available on the properties of harmonic functions. If proofs of some of the main results were also appended, the value of the book as one of reference would have been increased but, then, the size of the pamphlet would probably have exceeded the limit usually set for publications in this series.

V. GANAPATI IYER.

H. SUBRAMANI AIYAR. *A Text-book of Astronomy*, Chitra Publishing House, Trivandrum.

We are glad to welcome a text-book on Astronomy from an Indian author, written to meet the requirements of B. A. and Honours classes. The book is a happy combination of the descriptive and mathematical aspects both treated with exemplary clarity ; we have no doubt that it will prove very successful as a text-book, and will find special favour with the students and teachers of the Honours classes, the syllabus for which it covers pretty thoroughly. There is a large collection of examples at the end. We congratulate the author who is very well equipped for writing a work in Astronomy, on this useful production.

R. V.

OYSTIEN ORE, *L'Algebre Abstraite*, Actualités Scientifiques et Industrielles, No. 362.

The average mathematical teacher of the Indian University can have hardly any idea of the extent to which Abstract Algebra (or *Modern Algebra*) has developed in this century. An idea of the scope and the generality which it has attained, and of its power and elegance can be had by going through the pages of the excellent treatise of Van der Waerden on '*Modern Algebra*' in the Julius Springer series, or of Otto Haupt's *Einführung iu die Algebra* (2 Vols.) (Leipzig, 1929.) In this booklet of 52 pages, Professor Ore sketches the main concepts and the crowning results of the theories of modern Algebra. The topics discussed are : Group, Rings, Corpus, Homeomorphism, Isomorphism, algebraic extension of a corpus, the theory of Galois, Ideal-theories in Commutative rings, Hypercomplex numbers. Some of the more recondite topics, e.g. topological corpuses are briefly touched upon. The references given at the end are mostly to classical treatises. As an elegant bird's eye view of the subject, the book is very useful.

R. V.

RENE DE POSSELL, *Sur la theorie mathématique des Jeux de Hasard et de Reflexion*. Actualités Scientifiques et Industrielles No. 436, Hermann & Co., pages 40, price 10 francs.

This book is devoted to the indication of an interesting analysis of games, on the lines of Von Neumann "Uber die Gesellschaftspiele" *Mathematische, Annalen*, tome 100, 1927, and E. BOREL *Eléménts de la théorie des probabilités*. Games are classified according to the three elements of Reflexion, Hasard and Ruse ('Ruse' referring to the alteration of play due to the knowledge of the adversary's thought). The typical example of the '*Jeu de Reflexion*' is the one played by youngsters, in which two persons add alternately numbers less than 10 to an initially chosen number, the winner being the one who first reaches 100.

The theory of this kind of game and its extensions are studied in the book, and several interesting theorems are proved. The succeeding section treats of games of chance typical of which is the roulette. The last section analyses the 'society game' in which all the three factors of Reflexion, Hasard and ruse are present, and closes with a theorm of Von Neumann in which the influence of ruse is eliminated.

R. V.

MACLEAN, John: *Descriptive Mathematics*, Macmillan & Co., Bombay 1935, Pp. xvi+143; Price Rs. Two and Annas Eight.

To its disciples Mathematics presents two aspects; one that of a powerful intellectual instrument—perhaps the most powerful after "language" and the means of its recording—that the wit of man has devised to solve problems arising from his contact with the external world, and the other that it is the "home of the free creative spirit" where the Artist-Mathematician is subject only to the limitation of his common humanity. This blend of idealism and practicality which run like the warp and the woof through the mathematical fabric, has saved it on the one hand from the barenness which awaits too exclusive a stress on the needs of the moment, and on the other hand from getting itself lost in the trackless spaces of its own freedom.

The conscientious teacher effects a synthesis of both these aspects in his own teaching, now seeking by a heuristic approach to give his pupils something of the joy of the adventurer in uncharted regions of thought, now taking them round the mathematical showroom where the highly finished articles are available, each at its own price in the shape of intellectual honesty and mental effort.

The book under review effects a balanced synthesis at somewhere near the Intermediate in Arts level. The reader gets a panoramic view of the wide domain where mathematical ideas have sway, and glimpses of outlying regions beyond. He is taught to be bold, alert and critical and is constantly reminded that mathematics will not prevent him from making a fool of himself if he does not use his common sense. He learns the technique of various processes and perhaps also something of the way in which he could devise others himself. And lastly he is put on his mathematical good behaviour and is asked to follow a moral code which includes precepts like the following: "Follow a definite plan giving letters to your note-books and numbers to their pages; make your own system of indexes and cross references." "Ask yourself what accuracy is worth while and has a meaning". "If you fall into doubt as to how to proceed, discover a proper method by trying a very simple example as similar as possible to what you are attempting" etc. The book itself abounds in applications of these maxims in the discussion of concrete problems

The function concept as the basic concept of Mathematics receives a generous share of attention. The student is encouraged to carry out extensive arithmetical computations and tabulation with the help of the slide rule and various devices like detached coefficients, the use of finite differences and interpolation. Functional relationship is studied by a variety of methods—graphic representation (Cartesian, Polar, Triangular), functional scales, nomograms, contour maps. The functions of elementary analysis *viz.* polynomials trigonometric, exponential and logarithmic functions are studied and also the processes of differentiation and integration. There are chapters dealing with probability, frequency tables, histograms etc.

The wealth of material and the breeziness of the treatment mark such a wide departure from the staid and sober courses of our Universities that there must be considerable divergence of opinion regarding its use as a text-book. The author has succeeded in persuading his own University of Bombay to

allow its Intermediate in Arts candidates to offer Statistics and Nomograms as an alternative to the first paper on Analytical Geometry and Solid Geometry, and there can be no doubt that for those whose mathematical education does not extend further, this should be of far greater value than the course it replaces. Mathematical education in India is badly in need of fresh ideas and the experiment will be watched with interest. In any case, the reviewer would love to see instituted as an integral part of a student's mathematical education, a course of work in a mathematical Laboratory somewhat along the lines advocated in Prof. Macleans highly stimulating book. Such a course would introduce a breath of fresh mountain air into the somewhat stuffy atmosphere of our lecture rooms. But, are the windows open ?

A. NARASINGA RAO.

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## BOOKS RECEIVED FOR REVIEW

M. LAVRENTIEFF : *Sur les fonctions d'une variable complexe représentables par des series de polynomes*, Actualités Scientifiques et Industrielles No. 441, Hermann and Co., 1936, Price 15 fr.

DENJOY, Armand : *Introduction à la théorie des fonctiones de variables réelles,* Parts I and II Actualités Scientifiques Nos. 451, 452, 1937, Price 12 fr. each.

SARKAR, Madhusudam and DATTA, Binode Behari : *Astronomy*, Kamala Book Depot, Calcutta, Price Rs. 5.

GUNJIKAR, K. R. *An Introduction to the Calculus,* Part I, Differential, Oxford Univ. Press, 1937, pp. 182.

LARMOR, Sir Joseph (Edited by): *Origins of Clerk Maxwell's Electric ideas* as described in familiar letters to William Thomson, Cambridge 3/6.

DUNNINGTON : *Carl Friedrich Gauss*, Louisiana State University Press, 1937, pp. 90, $ 1·00.

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# ANNOUNCEMENTS AND NEWS

The authorities of the Lucknow University have been pleased to invite the Indian Mathematical Society to hold its Tenth Conference at Lucknow during the third week of March 1938. The Committee of the Indian Mathematical Society has gratefully accepted the invitation. Papers intended for the Conference may be sent to Prof. A. Narasinga Rao, Annamalainagar, along with a brief abstract so as to be received before the 31st January 1938.

Prof. G. N. Watson, Prof. E. H. Neville and Dr. R. Vaidyanathaswami have very kindly accepted the invitation of the Indian Mathematical Society to become Honorary Members of the Society.

Dr. R. Vaidyanathaswami and Dr. Ram Behari have been elected Secretaries of the Indian Mathematical Society for 1937-38.

The following gentlemen have been elected members of the Society:

Badri Nath Chadha Esq., M.A., B.T., Govt. Intermediate College, Pasrur (Sialkote District).

D. S. Misra Esq., M.Sc., Research Scholar, University of Lucknow, Lucknow.

N. S. Nagendranath Esq, M.Sc., Lecturer in Applied Mathematics, Andhra University, Waltair.

N. Venkoba Rao Esq., M.Sc., Assistant Master, S. L. N. High School, Bangalore.

V. S. Venkatanarayana Esq., B.A. Hons., Nellore.

K. Rangaswami, M.A., M.Sc., Lecturer, Madura College, Madura.

The Osmania University has appointed a sub-committee of the Research Board to make recommendations regarding the award of research degrees in all subjects including Mathematics. The Annamalai University has passed regulation re the Ph.D. degree which is to be awarded on a thesis to be submitted after 3 years of research done in the University for Honours candidates and 2 years in the case of those who have taken the M.Sc. degree.

Among the British Scientists interested in Mathematics and Mathematica Physics who will be visiting Calcutta to participate in the Jubilee Session of the Indian Science Congress will be the Rt. Hon. Lord Rutherford of Nelson, Sir Arthur Eddington, Dr. F. W. Aston, Prof. G. P. Thomson and Prof. E. J. M. Stratton. The British delegation is expected to arrive at Bombay in the middle of December and to visit places of importance in North India *viz.* Sanchi, Agra, Delhi, Dehra Dun, Benares before reaching Calcutta about the end of December. After the conference, the delegation will return to Bombay *via* Madras and Bangalore.

Dr. R. A. Fisher has kindly accepted the invitation to visit the Mysore University in January 1938.

B. R. Seth, M.A., Ph.D., Lecturer in Mathematics, Hindu College, Delhi, was awarded the D. Sc. Degree by the London University last June. The subject of his thesis was Elasticity. He will return to join his post in the Hindu College in October, after the holidays.

Mr. N. S. Nagendranath, M.Sc., who was recently appointed lecturer in Applied Mathematics in the Andhra University, is proceeding to Cambridge on an "1851 Exhibition Fellowship" tenable for 2 years for research in Nuclear Physics. We offer our hearty congratulations to Mr. Nagendranath who is the first Indian to win this Fellowship. Mr. Nagendranath will be attending the symposium on Theoretical Physics to be held at Copenhagen in September 1937.

Mr. Akbar Ali, M.A., Lecturer in Mathematics, Osmania Intermediate College, Gulburga is proceeding to England for higher studies in Mathematics. Mr. Raghavendra Rao, M.A. (Osmania) has been appointed lecturer in his place.

Mr. K. Rangaswami, M.Sc. has been appointed Lecturer in Mathematics in the Annamalai University in place of Mr. C. V. Lobo who has been taken over into Government Service.

We regret very much to record the death on 8-5-1937 of Dr. S Mukhopadhyaya, retired Professor, Calcutta University. Dr. Mukhopadhyaya served as a member of the committee of the Indian Mathematical Society and was President of the Calcutta Mathematical Society for 1937. He has made valuable contributions to Geometry and his collected geometrical papers have been published by the Calcutta University.

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## GLEANINGS

Not the Eye but the Spirit furnishes the proof of theories.

EINSTEIN

54. What is the nature of the problems confronting the human race for which a method is required? The problems which dire necessity sets before us are not solved by lengthy reflection; for unforeseen accidents we require unforeseen expedients, foresight for apprehended dangers. And the Method has as much to furnish the problem as to elicit the answer. *Verily in the clear statement of the problem lies more than half its solution.* How this familiar phrase confutes the attitude of those who put forward, as the one and only principle of experiment, *the putting the ear to nature*, without any idea what to listen for!

W. H. YOUNG,
*The mathematical method and its limitations:*
Bologna Math. Congress.

55. We may go farther and say that *the life of our mind* consists, above all in a germination of new problems, each of which suggests another. There is no actual *solving* of problems, but a *referring back* from the more to the less complex, or *vice versa* according to the trend of our thoughts. We cannot even claim to judge the unknown by the known, but only the less known by the more familiar.

W. H. YOUNG,
*The mathematical method and its limitations:*
Bologna Math. Congress.

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# SOLUTIONS TO QUESTIONS

## Question 1718

(V. RAMASWAMI IYER AND A. NARASINGA RAO):—Given a figure consisting of five lines in a plane and their Miquel-circle; show that if one of the lines be rubbed out, it can be restored by means of circle-and-line constructions.

*Solution by T. K. Raghavachari*

We have the following classical theorem of Miquel-Clifford:

With any even number of lines in a plane is associated a point called their Miquel point, and with any odd number is associated a circle, called their Clifford circle. If the number of lines in a set is an odd number $m$, the omission of these lines one at a time from the set gives $m$ subsets each of which is even, and the Miquel points of these subsets are concyclic on the Clifford circle of the original set. If the number of lines in a set is an even number $n$, the omission of these lines, one at a time, gives $n$ subsets each of which is odd, and the Clifford circles of these subsets are concurrent in the Miquel point of the original set.

Let us take an odd number $(2n+1)$ of lines in a plane. Let us call the lines $l_a$, $l_b$, $l_c$ etc. Let the Clifford circle $(C_0)$ of this set of lines be given and the line $l_x$ then rubbed out. Let us try to restore the line $l_x$ by circle-and-line constructions. We shall denote the Clifford circle associated with the set of $(2n-1)$ lines obtained by omitting, say, $l_p$, $l_q$ from the given set by $C\,(pq)$, and the Miquel point of the set obtained by omitting $l_p$ will be denoted by $M\,(p)$ etc. We can construct with the help of ruler and compass, the C-circles $C\,(xa)$, $C\,(xb)$, $C\,(xc)$, $\cdots$ ; and we know the point $M\,(x)$ by the same circle-and-line constructions. The circle $C\,(xr)$ cuts the given circle $C_0$ (associated with the set of $2n+1$ lines) in $M\,(x)$ and $M\,(r)$. Thus we obtain all the $(2n+1)$ points M on $C_0$. We know by similar constructions all the points $M\,(x\cdots)$ and the circles $C\,(x\ldots.)$. (The only thing to be noted is that the names all involve an $x$ in the suffix, at this stage). The circle through $M\,(p)$, $M\,(q)$ and $M\,(xpq)$ which we are now able to construct is the Clifford circle $C\,(pq)$. This is at once evident from the theorem quoted above. Now it is easy to see that the problem of restoring the missing line to the set of $2n$ lines, given the C-circle of the $(2n+1)$ lines, has been reduced to that of restoring the same missing line to the set of $(2n-2)$ lines (obtained by omitting two of the given lines from the previous set),

given the C-circle of these $(2n-1)$ lines. $(2n-2+\text{the}$ missing line). Proceeding in this manner, we ultimately come to the easiest "problem" of finding the third side of a triangle given in position two sides and the circum-circle: for, the Miquel point of two lines is their point of intersection and hence the Clifford circle of three lines is the circum-circle of the triangle formed by them.

Thus we see that given a figure consisting of an odd number $2n+1$ of lines in a plane, and the Clifford circle associated with them, if one of the lines be then rubbed out, we can restore it by circle-and-line constructions. The problem in the question is the particular case $n=2$.

*Remarks by A. Narasinga Rao*

When the late Mr. V. Ramaswami Aiyar communicated this problem to me in December 1935 and invited my solution for it, I drew his attention to the paper by Prof. Neville (On the inverse of the Miquel-Clifford Configuration, *Jour. Ind. Math. Soc. 1926.*) In this it is shown that if the straight lines generating the configuration are replaced by concurrent circles by inversion, the configuration becomes symmetric in the sense that it can be generated from the set of circles which meet at any of the points of multiple incidence in the configuration in exactly the same way as it was from the concurrent circles which are the inverses of the straight lines. Hence given the Clifford circle $C_0$ of $2n+1$ lines and also $2n$ of the lines, we obtain the Miquel point M of the $2n$ lines (which lies on $C_0$), and also the Clifford circles of sets of $2n-1$ out of the $2n$ lines. We thus obtain through M a set of $2n+1$ circles consisting of these and of $C_0$ and the configuration may be generated from these. (For a more detailed discussion of the generation of the configuration from any set of concurrent circles, reference may be made to my paper on "The group of the Miquel-Clifford Configuration" *Annamalai University Journal* Vol. I, pp. 43-47.)

Mr. Ramaswami Aiyar in his reply stated that Prof. Neville's Theorem did not occur to him at all, and that his question came as a by-product of other investigations. He wrote that he was interested in the correlation between $l_5$ and the Clifford Circle of $l_1\, l_2\, l_3\, l_4\, l_5$ when we are given $l_1\, l_2\, l_3\, l_4$. He suggested that $l_5$ might be represented by P the foot of the perpendicular from M (the Miquel point of $l_1\, l_2\, l_3\, l_4$ on it and the Clifford circle of $l_1\, l_2\, l_3\, l_4\, l_5$ by the point Q on it diametrically opposite to M), and wanted the Cremona transformation connecting P and Q. Given $2n$ st. lines and their Miquel pt. M, I have investigated directly in circle-space the Cremona Transformation which carries an additional straight line (regarded as a circle through

the ideal "point at infinity" of the tetracyclic plane) into a circle through M viz. the Clifford circle of the $2n+1$ lines; and the results have been published in the *Proc. of the Camb. Phil. Soc.* Vol. 33, p. 21 *et seq.*

Mr. Ramaswami Aiyar's own line of thought in connection with the five-line problem seems to be as follows, as far as it may be gathered from one of his letters*:

"Let us have four given lines $l_1\, l_2\, l_3\, l_4$. Then for any point P we have a conic $\Gamma_p$ which passes through P and the feet of the perpendiculars from it on the given lines."

"I found as follows:

(1) If $\Gamma_p$ be a rectangular hyperbola, *the locus of P is a circle.*

(2) It can be readily seen that this circle passes through the circumcentre of each of the triangles $(l_1\, l_2\, l_3)$ $(l_2\, l_3\, l_4)$ etc.".

"Accordingly our circle is the centre circle of the 4 lines. We can also see that the circle passes through the Miquel point M of the 4 lines; for, the pedal line of M and the perp. through M to the pedal line constitute a rectangular hyperbola such as we have sought."

(3) "We get a property of the point of intersection of the centre circle and any of the circles $(l_1\, l_2\, l_3)$. If $P_4$ be the point of intersection with the circle $(l_1\, l_2\, l_3)$, the pedal line of $P_4$ w.r.t. the triangle $(l_1\, l_2\, l_3)$ is parallel to $l_4$. I deduced my construction from this."

"Let us have next a five-line $(l_1 \ldots l_5)$ whose Miquel points when the lines are taken four at a time, are $M_1 \ldots\ldots M_5$. The locus of points P for which a $\Gamma_p$ exists in regard to the five-line is the Clifford circle of the lines.—*This is Clifford's theorem.* We get as follows".

"There is *just one point* P for which $\Gamma_p$ is a rectangular hyperbola. Hence the 5 centre circles of the five-line all co-intersect at a $\Omega$ on the Clifford circle. [This theorem was given by Bhimasena Rao and myself in our joint paper entitled "On Miquel points and circles and centre circles of a system of lines. It comes out *in a different connection* here.]"

"If P P′ are points on the Clifford circle equidistant from $\Omega$ then $\Gamma_p$ and $\Gamma_{p'}$ have the same eccentricity subject to the convention that a hyperbola and its conjugate have the same eccentricity ......The eccentricity of $\Gamma_p$ will be a minimum when P is the point diametrically opposite to $\Omega$ in the Clifford circle".

* The matter which follows consists of verbatim extracts except that the word "Miquel Circle" has been changed into "Clifford circle".

## Questions 1722 and 1723

(A. RANGANATHA RAO):—The director circle of the $\phi$-conic of two rectangular hyperbolas passes through their centres.

(A. RANGANATHA RAO):—The $\phi$-conic of two parabolas is concentric with the centre-locus of the pencil of conics determined by the two parabolas.

*Solution by T. K. Raghavachari*

Consider the four-point system of conics $S + \lambda S' = 0$ where

$$S \equiv ax^2 + by^2 + cz^2 = 0\ ;\ S' \equiv a'x^2 + b'y^2 + c'z^2 = 0.$$

Taking the co-ordinates to be areal, the centre of $S + \lambda S' = 0$

$$x(a + \lambda a') = y(b + \lambda b') = z(c + \lambda c')$$

Eliminating $\lambda$ from these equations, we get the equation of $\Gamma$, the centre-locus of the four-point system:

$$\Gamma \equiv yz(bc' - b'c) + zx(ca' - ac') + xy(ab' - a'b) = 0 \ldots (1)$$

The $\phi$-conic of S and S′ has the point equation:

$$\phi \equiv x^2/(bc' + b'c) + y^2/(ca' + c'a) + z^2/(ab' + a'b) = 0$$

The centre of the $\phi$-conic has, therefore, co-ordinates proportional to $(bc' + b'c)$, $(ca' + c'a)$, $(ab' + a'b)$.

The polar of this centre w.r.t. $\Gamma$ is the straight line

$$x[(ab' - a'b)(ca' + c'a) - (ca' - ac')(ab' + a'b)] + y(\quad) + z(\quad) = 0$$

*i.e.*, $$2aa'(bc' - b'c)x + \cdot\cdot\cdot + \cdot\cdot\cdot = 0$$

*i.e.*, $$x\left(\frac{1}{bc'} - \frac{1}{b'c}\right) + y\left(\frac{1}{ca'} - \frac{1}{c'a}\right) + z\left(\frac{1}{ab'} - \frac{1}{a'b}\right) = 0 \qquad \ldots \quad (2)$$

This straight line is the same as

$$\begin{vmatrix} x & y & z \\ 1/a & 1/b & 1/c \\ 1/a' & 1/b' & 1/c' \end{vmatrix} = 0$$

which is the line joining the centres of the conics S, S′.

Thus we have proved that the centre of the $\phi$-conic of two conics S, S′ is the pole of the line joining the centres of S, S′ with respect to the centre-locus of the four-point system determined by S and S′. [Madras University B.A. Hons. 1937.]

When the conics S, S′ are rectangular hyperbolas, $\Gamma$ becomes the circum-circle of the common self-polar triangle ABC which is also self-polar for the $\phi$-conic of S and S′. By Gaskin's theorem, $\Gamma$ cuts the director-circle of the $\phi$-conic of the rectangular hyperbolas S, S′ orthogonally. Therefore, by the theorem just proved, it follows that

the points of intersection of $\Gamma$ and the director-circle of $\phi$ are the centres of S and S′—which is question 1722.

That the $\phi$-conic of two parabolas and the centre-locus of the pencil determined by the parabolas are concentric is only a particular case of the general theorem proved. For in this case the centre of the $\phi$-conic and the line at infinity are pole and polar w. r. t. the $\phi$-conic as well as the centre-locus.

*Also solved analytically by S. Narayanan.*

## Question 1742

(K. RANGASWAMI)—Through the mid-point E of the perpendicular CD, drawn from the right angle C of a right-angled triangle ABC on AB, lines are drawn parallel to the sides of the triangle.

Show that:

(i) The six points of intersection of those lines with the sides of the triangle lie on a circle.

(ii) As C moves on the circumcircle of ABC, the centre of the circle through the six points describes an ellipse whose eccentricity is ½. Investigate the positions and lengths of the axes of the ellipse.

*Remarks by A. A. Krishnaswami Ayyangar*

Since CD is anti-parallel both to CB and CA, E its mid-point is the symmedian point of the triangle and hence the six points of intersection in (i) lie on the well-known Triplicate Ratio Circle or Lemoine circle whose centre is the mid-pt. of the line joining E to the circum-centre of ABC, viz. the mid-pt. of AB.

As C moves on the circumcircle of ABC, the locus of E is obviously an ellipse of which AB is the major-axis in position and magnitude and the minor axis is AB/2. Hence the centre of the Lemoine circle traces a similar and similarly situated ellipse, with the homothetic centre at the mid-pt. of AB. Its major axis is AB/2 and minor axis AB/4. *Hence the eccentricity is not ½ as wrongly stated in the question. It is* $\frac{\sqrt{3}}{2}$.

A direct elementary geometrical proof is fairly obvious and may be left to the reader.

*Also solved by another method, by G. Srinivasan M. V. Vaidyanatha Sastri and Hansraj Gupta.*

# QUESTIONS FOR SOLUTION

**1744.** (A. A. KRISHNASWAMI AYYANGAR)—If $a/b$ and $c/d$ be two positive fractions such that $ad - bc > 1$, shew that there exists at least one intermediate fraction with denominator less than $b$ or $d$ whichever is greater.

(A direct elementary proof is required not based on the properties of Farey's series).

**1745.** (D. R. KAPREKAR) We have

$$91 \times 819 = 74529$$

and also

$$\underline{9}9\underline{0}1 \times \underline{9}8\underline{0}1\underline{9}9 = \underline{9}7\underline{0}4\underline{9}5\underline{0}2\underline{9}9$$

where the corresponding numbers in the second multiplication are obtained by inserting 9 and 0 alternately before the first digit and between the successive digits of all the numbers in the first multiplication. (The inserted figures are shown underlined). It will be found that the multiplication is still true if we insert alternately blocks of 99 and 00 or 999 and 000 and so on.

It is required to find the principle underlying the result and to find other number pairs like 91 and 819 with the same property.

**1746.** (S. PANKAJAM)—If $x$ and $y$ be any two given integers, and $r$ an unknown integer, find ($a$) the integers $r$, and ($b$) the number of such values of $r$, for which

(i) $\binom{x}{r}$ is odd. (ii) $\binom{x}{r} - \binom{y}{r}$ is odd.

(iii) $\binom{x}{r} + \binom{x}{r-2}\binom{y}{2} + \binom{x}{r-4}\binom{y}{4} + \cdots$ is odd.

[ $\binom{x}{r}$, etc. are the usual binomial coefficients ].

**1747.** (K. RANGASWAMI)—Solve the following equations for $u, v, w$:—

$$\begin{aligned} u(m-n)^2 + vn^2 + wm^2 &= \lambda p^2 \\ un^2 + v(n-l)^2 + wl^2 &= \lambda q^2 \\ um^2 + vl^2 + w(l-m)^2 &= \lambda r^2 \end{aligned}$$

**1748.** (N. DURAIRAJAN)—Show that the locus of a point P in the plane of a triangle ABC such that the circles PAB, PAC are equal consists of a circle and a rectangular hyperbola.

**1749.** (B. RAMAMURTI)—Prove the following:—

There is a unique line intersecting three tangents of a rational norm curve $R^4$ in a space [4] of four dimensions. Through any point, in general, there will not pass such a line. If there be one, there must be $\infty^1$ such lines.

There is a unique plane intersecting four tangents of a rational norm curve $R^6$ in a space [6] of six dimensions. Through any point, in general, there will pass at least one such plane. If there be more than one there must be at least $\infty^2$ such planes.

Extend these results to higher dimensions.

**1750.** (C. N. SRINIVASIENGAR)—Prove the following properties of scrolls:—

(1) The tangent lines of a scroll at points along a generator and at right angles to it generate a rectangular paraboloid, which is closely associated with the locus of normals at points along the generator.

(2) The line of striction of either system of generators of these paraboloids is a straight line.

(3) The osculating quadric along a generator of a scroll always touches the scroll "internally".

(4) The osculating quadric along any generator of a right helicoid is a rectangular paraboloid.

(5) If at points along a generator of a general scroll tangent lines are drawn to the orthogonal trajectories of the curved asymptotic lines, these lines generate a quartic scroll having the given generator as a triple line. The quartic scroll reduces to a cubic in special cases e.g. for a right conoid (excluding the right helicoid).

**1751.** (K. NAGHABHUSHANA RAO)—Prove directly using the definition of an Integral Invariant Poincare's result that,

$\iiiint \sum \delta q_i \; \delta p_i \; \delta q_j \, \delta p_j$ where the summation is extended over the $\frac{1}{2}\, n(n-1)$ combinations of the indices $i$ and $j$, is an integral invariant of any Hamiltonian System in which $(q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n)$ are the variables.

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# THE

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*and others.*

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# ERRATA FOR VOLUME IV

| *Page* | *line* | *for* | *Read* |
|---|---|---|---|
| 38 | 33 | "linear of the type" | "linear equations of the type" |
| 41 | 10 | | omit full stop after "space" |
| 55 | 7 | $(r_b - r)^2$ | $(r_b + r)^2$ |
| 55 | 7 | $(r_c - r)^2$ | $(r_c + r)^2$ |
| 58 | 10, 11, 12 | Replace these last 3 lines of Thébault's paper by the following: | |

The point K″≡H′ for, the vertices B′, C′ coincide with A of the triangle ABC on the circumcircle of the triangle; the sides A′B′, A′C′ of the pedal triangle coincide with the altitude AA′ of ABC; the third side B′C′ becomes the tangent at A to the circle ABC; the ortho-centre H′ of A′B′C′ is the point of intersection of the parallels to BC and to OA through the points A, A′ or the *symmétrique* of O with respect to K.

| *Page* | *line* | *for* | *Read* |
|---|---|---|---|
| 73 | 25 | $\Sigma a_r\ x_r\ x_s$ | $\Sigma a_{rs}\ x_r\ x_s$ |
| 75 | 7 | plane | plano |

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